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Mapping Spaces, Configuration Spaces and Gauge Theory

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To my Parents
for their love and support.

Ταύτας τοίνυν τὰς τῶν μαθημάτων ἡδονὰς ἀμίκτους τε εἶναι λύπαις ῥητέον καὶ οὐδαμῶς τῶν πολλῶν ἀνθρώπων ἀλλὰ τῶν σφόδρα ὀλίγων.

Plato, *Φίληβος*, 52B

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

The present thesis considers the space of connections modulo based gauge equivalence on a principal $SU(2)$ bundle over a closed simply-connected smooth four-dimensional manifold M. Up to homotopy equivalence, this is the space of basepoint-preserving maps from M to $BSU(2)$, the classifying space of $SU(2)$. It depends only on the homotopy type of M, which is characterized by the intersection form. The $\mathbb{Z}/p\mathbb{Z}$-homology of the mapping space for p a prime not equal to 3 is computed and given in terms of the data associated to the intersection form. For the prime 3, partial results are obtained. The main method is to consider a fibration associated to a CW decomposition of M and to show that the corresponding Eilenberg-Moore spectral sequence collapses. These results generalize from manifolds to spaces homotopy equivalent to a bouquet of 2-spheres with a single 4-cell attached.

For the possible homotopy types the space of connections modulo gauge equivalence can attain, a classification is obtained in the following sense. The homotopy type of this space is uniquely determined by the rank, type and signature modulo *eight* of the intersection form. On the other hand, the homotopy type determines the rank, type and signature modulo *four* of the intersection form. Both results together give a complete classification for the case of spin manifolds. The homotopy types of the spaces of connections modulo gauge equivalence over two spin manifolds agree if and only if the intersection forms are of the same rank. These results use a classification of unimodular bilinear forms over the ring $\mathbb{Z}/4\mathbb{Z}$.

In a final part, a map is constructed from the labelled configuration spaces of points in the manifold to the mapping space. This map is shown to be asymptotically surjective in homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients. For homology with general coefficients, classes are constructed which are not approximated by this map.

Chapter 1

Introduction

Gauge theory, which has its origin in theoretical physics, is for a mathematician the theory of connections modulo gauge equivalence on a principal bundle. In Yang-Mills theory one considers the space $\mathcal{A}_P$ of the connections on a principal G-bundle P over a connected, smooth four-dimensional manifold M. In general G could be any compact Lie group, but for us, G will always be $SU(2)$. The bundle P is then determined by its second Chern class and we write $\mathcal{A}_k$ for $\mathcal{A}_P$, where k is the Chern number of P. Motivated by physics, one is interested in the minima of the Yang-Mills functional

$$\mathcal{YM}: \mathcal{A}_k \longrightarrow \mathbb{R}.$$

They can be characterized as certain solutions of a system of partial differential equations, the Yang-Mills equations. Among these, the self-dual or anti-self-dual solutions are of special interest. They form the so-called space of 'instantons' of 'charge' k.

On $\mathcal{A}_k$ there is the action of the gauge group $\mathcal{G}_k$ and one can consider the space of orbits $\mathcal{A}_k/\mathcal{G}_k$. It is more convenient to consider the action of the group $\mathcal{G}_k^\bullet$ of based gauge transformations, which acts freely on $\mathcal{A}_k$. The Yang-Mills functional is gauge-invariant, so we can define $\mathcal{M}_k$ to be the orbit space of instantons of charge k under the action of the based gauge group

$\mathcal{G}_k^\bullet$: this is the 'moduli space of self-dual instantons'.

Let us now assume that the underlying manifold M is closed (that is compact without boundary) and simply-connected. While $\mathcal{A}_k$ is an affine space (and hence contractible), there is a rich topological structure on the quotient $\mathcal{A}_k/\mathcal{G}_k^\bullet$. The inclusion of the subspace of self-dual connections induces a map

$$i : \mathcal{M}_k \longrightarrow \mathcal{A}_k/\mathcal{G}_k^\bullet.$$

Topologically, this map was first examined by M. Atiyah and J. D. S. Jones (see [3]) who showed that, taking M to be the four-sphere, the map i is an asymptotic surjection in homology, i. e. the map induced by i in homology is surjective in a range that gets arbitrarily large for increasing values of k. The homotopy type of the target space $\mathcal{A}_k/\mathcal{G}_k^\bullet$ is independent of k and for the case of the four-sphere turns out to be that of pointed self-maps of the three-sphere of fixed degree, a space whose homology has been studied extensively. The proof is based on an approximation of this space by configuration spaces of sets of k points in $\mathbf{R}^4$ and a factorization of the approximation map through $\mathcal{M}_k$.

This result led to some interesting conjectures. Atiyah and Jones conjectured that the map i is in fact an asymptotic homology and homotopy equivalence. This was proved by C. Boyer *et al.* ([4]), who also gave an explicit lower bound on the range in which equivalence occurs depending on k. One could further conjecture that the same holds for more general four-manifolds. Results obtained by C. Taubes ([28]) imply that the surjectivity part holds for any compact connected four-manifold and for any compact simple Lie group G, but still many questions remain open. In particular Taubes' result does not give any estimate on the range of surjectivity.

Apart from being of independent interest, the homology of the spaces $\mathcal{M}_k$ can be used to obtain information about the underlying manifold M. This concept has been used with great success by S. K. Donaldson, whose results, in combination with work carried out by M. Freedman, led to the discovery of the existence of exotic differentiable structures on $\mathbf{R}^4$. The Donaldson polynomial invariants, able to distinguish differentiable structures on one and the same underlying topological manifold, are derived from the map induced by i in homology (see [11]).

It is maybe worth mentioning that, in very recent months, some of Donaldson's results have been reproduced, together with some new results, using complex line bundles over M, together with a *spin*^{c} structure. This approach originates from ideas of N. Seiberg and E. Witten [27], and at present rapid progress is being made in this field. For a taste of the arguments involved, see [18, 29].

The aim of the present thesis is to study the topology of the space $\mathcal{A}_k/\mathcal{G}_k^\bullet$ for a general compact, simply-connected four-manifold M. In general, this space has the (weak) homotopy type of the space of basepoint-preserving maps from M to $BSU(2)$, the classifying space of $SU(2)$, so its topology only depends on the homotopy type of M. We examine the homology and homotopy of the space $\mathcal{A}_k/\mathcal{G}_k^\bullet$, expressing our results as far as possible as 'functors' of the known invariants of M. Secondly, we address the question of how much topological information about the underlying manifold can be recovered from the homotopy type of the space of connections modulo gauge equivalence. Based on a result by G. Masbaum [20], we obtain restrictions on the number of possible homotopy types the space $\mathcal{A}_k/\mathcal{G}_k^\bullet$ can attain for different underlying manifolds. Combining these results with our homology calculations, we obtain partial results towards the classification of the homotopy types occurring as $\mathcal{A}_k/\mathcal{G}_k^\bullet$ for some manifold, or from the opposite point of view, results about how much of the topology of the manifold can be recovered. In particular our methods suffice to give a complete classification of the homotopy types occurring as $\mathcal{A}_k/\mathcal{G}_k^\bullet$ for a spin manifold. Motivated by the proof of Atiyah and Jones, we then give an approximation of the space $\mathcal{A}_k/\mathcal{G}_k^\bullet$ by labelled configuration spaces of points in the manifold and, using our homology calculations, show that it is asymptotically surjective in homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Even though the problem arises in gauge theory, all calculations are carried out using classical algebraic topology. For the background in gauge theory, see [12, 14].

The thesis is organized as follows. Chapter 2 sums up some well-known facts about four-manifolds. The oriented homotopy type of a compact, simply-connected four-manifold depends only on the intersection form. This is a classical result by J. H. C. Whitehead and J. Milnor (see[24]). In fact, isomorphism classes of symmetric bilinear forms over $\mathbb{Z}$ are in one-to-one correspondence to oriented homotopy types of spaces homotopy equivalent to a bouquet of 2-spheres with a single 4-cell attached, and much of our further calculation is carried out in the more general context of these spaces. We recall the proof in section 2.1. The remainder of the chapter contains a list of elementary results on the spaces considered, mainly in order to conveniently refer to them later.

Chapter 3 briefly reviews the theory of symmetric bilinear forms over the integers (section 3.1) and over the finite fields $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ (section 3.2) as far as required in the following chapters. All this is well known and contained in the standard literature on the subject. We also require a classification of unimodular bilinear forms over the ring $\mathbb{Z}/4\mathbb{Z}$, which we give in section 3.3. We have not been able to find this in the literature. There exists, however, a very similar classification for mod four valued quadratic forms over a $\mathbb{Z}/2\mathbb{Z}$ vector space (see [32]).

As mentioned earlier, it is well known that the space $\mathcal{A}_k/\mathcal{G}_k^\bullet$ has the homotopy type of $\mathrm{Maps}(M, BSU(2))$. We give some details in section 4.1. The latter space is the homotopy fibre of the map

$$\mathrm{Maps}(X, BSU(2)) \xrightarrow{f^\sharp} \mathrm{Maps}(S^3, BSU(2))$$

where X denotes the 2-skeleton of M and the map $f^\sharp$ is determined by the attaching map of the 4-cell of M to X. This fibration is the key to our calculations in homotopy and homology, contained in chapters 4 and 5. The homology of the spaces $\mathrm{Maps}(X, BSU(2))$ and $\mathrm{Maps}(S^3, BSU(2))$ is well known. The homotopy groups of these spaces are, in general, not known, but they can be easily expressed in terms of the homotopy groups of S^3. The map induced by $f^\sharp$ on homotopy groups depends only on the rank and type of the intersection form of M. As a consequence, the homotopy groups of $\mathrm{Maps}(M, BSU(2))$ only depend (up to isomorphism) on these two data, which, in turn, can be recovered from the first two homotopy groups of the mapping space. These calculations, which later form the basis for the computations in homology, are contained in section 4.2. In section 4.3 we briefly recall the calculation of the homology of $\mathrm{Maps}(X, BSU(2))$. It carries a natural Hopf algebra structure and is naturally isomorphic, as a Hopf algebra, to a polynomial Hopf algebra primitively generated by the second cohomology of M.

Given the space X, all 4-dimensional complexes with one 4-cell, having X as 2-skeleton, are classified by the attaching maps $f \in \pi_3(X)$. The map that sends f to $f^\sharp\colon \mathrm{Maps}(X, BSU(2)) \to \mathrm{Maps}(S^3, BSU(2))$ gives a group homomorphism

$$\sharp\colon \pi_3(X) \longrightarrow [\mathrm{Maps}(X, BSU(2)), \mathrm{Maps}(S^3, BSU(2))]$$

where square brackets denote the set of homotopy classes, with its group structure inherited from the loop space $\mathrm{Maps}(S^3, BSU(2))$. In section 4.4

we establish a natural factorization of $\sharp$ restricted to the subgroup of $\pi_3(X)$ corresponding to even intersection forms, which will later be useful for the calculations in homology. As a first corollary, we obtain the kernel of the map $\sharp$. This kernel was first determined by G. Masbaum (see [20]). We give a proof that stays completely within the topological category.

Our main result of chapter 4 is obtained by combining the knowledge about the kernel of $\sharp$ with some algebraic calculations about bilinear forms: If the intersection forms of M_1 and M_2 are equivalent over $\mathbf{Z}/12\mathbf{Z}$, then the spaces $\mathrm{Maps}(M_1, BSU(2))$ and $\mathrm{Maps}(M_2, BSU(2))$ are homotopy equivalent. If M_1 and M_2 are manifolds, so their intersection forms are unimodular, this result, together with the classification of unimodular bilinear forms of chapter 3, implies that a sufficient condition for the existence of a homotopy equivalence between the mapping spaces is that the intersection forms of M_1 and M_2 have the same rank and type and their signatures are (up to sign) congruent modulo eight. In particular, if M_1 and M_2 are spin manifolds, the signatures are always divisible by eight, so the homotopy type of the mapping space just depends on the rank of the intersection form.

Chapter 5 contains the calculations in homology. Our principal tool is the Eilenberg-Moore spectral sequence of the fibration

$$\mathrm{Maps}(M, BSU(2)) \longrightarrow \mathrm{Maps}(X, BSU(2)) \xrightarrow{f^\sharp} \mathrm{Maps}(S^3, BSU(2)).$$

Our main result is that this spectral sequence collapses for homology with coefficients in $\mathbf{Z}/p\mathbf{Z}$ for p a prime not equal to 3. As a consequence, the $\mathbf{Z}/p\mathbf{Z}$ homology of the mapping space is additively given as

$$\mathrm{Cotor}^{H_*(\mathrm{Maps}(S^3, BSU(2)))}\Big(H_*\big(\mathrm{Maps}(X, BSU(2))\big), \mathbf{Z}/p\mathbf{Z}\Big).$$

Some details about the Eilenberg-Moore spectral sequence, the precise definition of the Cotor functor and some methods to calculate it are given in section 5.1. All this, of course, is well known, as are some facts needed about the homology of loop spaces, which we include in section 5.2. In order to calculate the Cotor functor, we have to know the map induced by $f^\sharp$ in homology. Notice that the domain as well as the target space are Hopf algebras. Since the map $f^\sharp$ is not a map of loop spaces, the map induced in homology does not, in general, preserve the Pontryagin product. However, we prove that the induced map in $\mathbb{Z}/2\mathbb{Z}$ homology does preserve the Hopf algebra structure and this surprising fact, together with the homotopy calculations of section 4.2, allows us to determine the map in $\mathbb{Z}/2\mathbb{Z}$ homology and the corresponding Cotor functor. The collapsing of the spectral sequence is established separately for odd and even forms by comparing the Eilenberg-Moore and Leray-Serre spectral sequences of several fibrations. These calculations are contained in section 5.3. A preliminary version of these calculations appeared in the author's diploma dissertation [23], out of which the present work grew. For the case of even intersection forms, G. Masbaum also gives a calculation of the $\mathbb{Z}/2\mathbb{Z}$ homology of the mapping space (see [19]). He considers the Leray-Serre spectral sequence of the fibration

$$\mathrm{Maps}(S^4, BSU(2)) \longrightarrow \mathrm{Maps}(M, BSU(2)) \xrightarrow{f^\sharp} \mathrm{Maps}(X, BSU(2))$$

which he shows to collapse, if the intersection form of the space M is even. Our proof is more elementary in the sense that it does not make any appeal to infinite loop spaces or Bott periodicity. Also, his methods do not give any result for the case of odd intersection forms.

Localized at an odd prime $p \geq 5$, the map $f^\sharp$ becomes null-homotopic. This leads to the collapsing result of the spectral sequence for $\mathbb{Z}/p\mathbb{Z}$ homology. This result is a consequence of the knowledge of $ker\sharp$ and is due to Masbaum. We include it in section 5.4.

The behaviour in $\mathbb{Z}/3\mathbb{Z}$ homology is different from the other primes. In section 5.5, we compute the map induced by $f^\sharp$ in homology, which turns out not to preserve the Pontryagin product. We then show that the complete information about the $\mathbf{Z}/3\mathbf{Z}$ intersection form can be recovered from the cup product in the cohomology of the mapping space. The full computation of the $\mathbf{Z}/3\mathbf{Z}$ homology, however, has to be left to a future project. Nevertheless, in the case where M is a manifold, the additional information we recover from the $\mathbf{Z}/3\mathbf{Z}$ homology can be used to compute the determinant of the intersection form, which in turn allows us to recover the signature modulo four (up to sign) of the intersection form.

Section 5.6 sums up all the information about the intersection form of a manifold that can be recovered from the homotopy and homology of the mapping space, i. e. from the space of connections modulo based gauge equivalence. This information translates into necessary conditions on manifolds for the associated mapping spaces to have the same homotopy type. On the other hand, in section 4.4 we obtained conditions which are sufficient for the homotopy types of the associated mapping spaces to be the same. As mentioned earlier, for spin manifolds these conditions agree, so we get a complete classification of the homotopy types that occur. For manifolds with odd intersection form, these conditions still leave certain cases undecided. However, for a given set of data as they are recoverable from the mapping spaces, there are at most two potentially different homotopy types matching these data, which we cannot distinguish.

In chapter 6 we construct an approximation of the mapping space by configuration spaces. In section 6.1 we start with some technical results about spaces of sections of some fibre bundles. In [22], D. McDuff defined certain configuration spaces associated to a smooth manifold M and showed that they approximate homologically the spaces of sections of certain bundles associated with the tangent bundle of M. It is mentioned in her paper how to generalize these results to labelled configuration spaces, but no details are given. In section 6.2 we therefore partly review her work, stating the changes to be made in our case, as we go along. In section 6.3 we give some details about the original construction used by Atiyah and Jones [3] to prove their approximation theorem. Let M_0 denote M with the basepoint removed. In section 6.4 we construct a map

$$\rho\colon C_k(M_0; FM) \longrightarrow \mathrm{Maps}_k(M, BSU(2))$$

where $C_k(M_0; FM)$ denotes the space of unordered points in M_0, with labels attached to each of them. Each label lies in the fibre of the frame bundle of the tangent bundle over the respective point. This map can be constructed as follows. An element in $C_k(M_0; FM)$ describes k points in M. Choose a small disk around each of these points. Define the map $M \to BSU(2)$ to send the complement of the disks to the basepoint and to be the standard degree 1 map $S^4 \to BSU(2)$ on each of the disks, using the framing coming from the label at each point to identify the small disk with the standard 4-disk. We use the material proved in section 6.2 to show that ρ is a well-defined continuous map. We then show that the map ρ induces an asymptotic surjection in $\mathbf{Z}/2\mathbf{Z}$ homology.

In section 6.5 we discuss the map ρ in cohomology with general coefficients. We identify a cohomology class $2Q \in H^4(\mathrm{Maps}_k(M, BSU(2)); \mathbf{Z})$ which is in the kernel of ρ^*. This class, which is related to the intersection form of the manifold, is also non-zero in $\mathbf{Z}/p\mathbf{Z}$ homology for $p \geq 5$ a prime. One can define maps ρ', π such that ρ factors as

$$C_k(M_0; FM) \xrightarrow{\rho'} \mathrm{Maps}(M, S^4) \xrightarrow{\pi} \mathrm{Maps}(M, BSU(2))$$

and finds that $\pi^*(2Q) \in H^4(\mathrm{Maps}(M, S^4))$ is already zero. Here the map π is induced by the map $S^4 \to BSU(2)$ coming from the embedding of $\mathbf{H}P^1$ in

$\mathbb{H}P^\infty$. More generally speaking, the filtration $\mathbb{H}P^1 \subset \mathbb{H}P^2 \subset \cdots \subset \mathbb{H}P^n \subset \ldots$ of $\mathbb{H}P^\infty$ gives rise to a filtration of $\mathrm{Maps}(M, \mathbb{H}P^\infty)$ and if there is an approximation of $\mathrm{Maps}(M, BSU(2))$ which is asymptotically surjective in homology with general coefficients, it cannot factor through the lowest term of this filtration. We show that it is enough to go one step further up in the filtration: The inclusion $\mathbb{H}P^2 \hookrightarrow \mathbb{H}P^\infty$ induces a map between the mapping spaces which is surjective in homology with $\mathbb{Z}/p\mathbb{Z}$ coefficients for any prime p.

Chapter 2

Four-Manifolds

2.1 Homology and Intersection Form

The aim of this section is to recall some well-known facts about closed four-dimensional manifolds, where 'closed' means compact with empty boundary. Let M be a connected, simply-connected, closed 4-manifold. It is well known that M has the homotopy type of a bouquet of 2-spheres with a single 4-cell attached. Let us, more generally, consider any space $Y \simeq e^4 \cup_f X$ where $X \simeq S^2 \vee \ldots \vee S^2$ and $f \in \pi_3(X)$. Let $H_*(Y)$ and $H^*(Y)$ denote integral homology and cohomology. Choose a generator $\mu \in H_4(Y)$. We will call such a generator an *orientation class* of Y. The *intersection form* of Y, denoted by $(\,\cdot\,,\,\cdot\,)$, is a symmetric integer-valued bilinear form on the free $\mathbf{Z}$-module $H^2(Y)$. It is defined by $(\alpha, \beta) = \langle \alpha \cup \beta, \mu \rangle$ for $\alpha, \beta \in H^2(Y)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the Kronecker pairing between cohomology and homology. If Y is a closed manifold, it follows from Poincaré duality that the intersection form is *unimodular*, i. e. the adjoint map $H^2(M) \to \mathrm{Hom}(H^2(M), \mathbf{Z})$ sending $\alpha \in H^2(M)$ to $(\,\cdot\,, \alpha)$ is an isomorphism. The intersection form is a powerful tool for the classification of 4-manifolds. The main aim of this section is to recall that for simply-connected closed 4-manifolds the oriented homotopy type is uniquely determined by the isomorphism class of the intersection form.

More generally, this holds for spaces $Y \simeq e^4 \cup_f X$ where $X \simeq S^2 \vee \ldots \vee S^2$. The homotopy type of Y is determined by the homotopy class $[f] \in \pi_3(X)$ of the attaching map of the 4-cell. For a $\mathbb{Z}$-module F denote by $\mathcal{S}^n(F)$ the n-fold symmetric power of F i.e. the quotient of $F^{\otimes n}$ by the action of the symmetric group Σ_n. The intersection form can be described as an element of $\mathrm{Hom}(\mathcal{S}^2(H^2(X)), \mathbb{Z})$. Consider the map

$$\gamma : \pi_3(X) \longrightarrow \mathrm{Hom}(\mathcal{S}^2(H^2(X)), \mathbb{Z})$$

that assigns to $[f] \in \pi_3(X)$ the intersection form of the complex $e^4 \cup_f X$ with *standard orientation*, which is defined to be the one that, by collapsing X, induces the standard orientation on S^4. One verifies the following.

Lemma 2.1 *The map γ is a linear isomorphism.* □

Corollary 2.2 *Let X_1 and X_2 have the homotopy type of $S^2 \vee \ldots \vee S^2$, and let $Y_1 = e^4 \cup_f X_1$ and $Y_2 = e^4 \cup_g X_2$ with intersection forms Q_1 and Q_2. There exists an orientation-preserving homotopy equivalence $Y_1 \to Y_2$ if and only if Q_1 and Q_2 are isomorphic.*

Proof: Clearly, if $Y_1 \simeq Y_2$ such that orientations are preserved, then Q_1 and Q_2 are isomorphic. Conversely, let $r : H^2(X_2) \to H^2(X_1)$ be an isomorphism such that $(r \otimes r)^*(Q_1) = Q_2$. By possibly reversing the signs of f and g, we can without loss of generality assume that Y_1 and Y_2 have standard orientation. Choose a map $h : X_1 \to X_2$ inducing r in cohomology and let Q be the intersection form of the complex $Y = e^4 \cup_{h \circ f} X_2$ with standard orientation. There is an orientation-preserving homotopy equivalence $\mathrm{id} \cup_f h : Y_1 \to Y$ and, for $\alpha, \beta \in H^2(Y)$, one obtains

$$Q(\alpha \otimes \beta) = Q_1(h^*(\alpha) \otimes h^*(\beta)) = Q_1(r(\alpha) \otimes r(\beta)) = Q_2(\alpha \otimes \beta)$$

so $Q = Q_2$. Using the isomorphism γ of lemma 2.1, we see that $\gamma(g) = Q_2 = Q = \gamma(h \circ f)$, so $g \simeq h \circ f$ and $Y \simeq Y_2$. Again, orientations are preserved by this equivalence. □

Remark: M. Freedman has classified closed simply-connected 4-manifolds up to homeomorphism. For every symmetric, unimodular bilinear form Q on a free $\mathbb{Z}$-module F there are at least one and at most two homeomorphism types of 4-manifolds realizing Q as their intersection form. If two homeomorphism types exist then they are distinguished by the *Kirby-Siebenmann invariant*, a $\mathbb{Z}/2\mathbb{Z}$-invariant that is the obstruction to the existence of a smooth structure on the cross product of the manifold with the real line. In particular, two smooth manifolds with isomorphic intersection forms are homeomorphic (see [15, 16]). On the other hand, S. K. Donaldson has given some obstructions for the smoothability of a topological 4-manifold and severe restrictions on the possible intersection forms of smooth 4-manifolds (see [9, 10, 11]).

If we choose a basis $\{\alpha_1, \ldots, \alpha_r\}$ for $H^2(X)$, the intersection form of X is described by a symmetric r by r matrix Q, where $Q_{ij} = (\alpha_i, \alpha_j)$. Notice that unimodular forms correspond precisely to matrices of determinant ± 1. Two forms are equivalent if and only if their matrices Q and Q' are related by the formula $Q' = A^T Q A$ for some invertible matrix A.

Example 2.3 For the complex projective plane $\mathbb{CP}^2$, with the standard orientation provided by the complex structure, we get the intersection matrix (1). As a CW complex, $\mathbb{CP}^2 \cong e^4 \cup_\eta S^2$, where η is the Hopf map. For $\overline{\mathbb{CP}^2}$, which denotes $\mathbb{CP}^2$ with the opposite orientation, we obtain the matrix (-1).

Example 2.4 For the manifold $S^2 \times S^2$ with the standard orientation, taking the basis dual to the two standard inclusions ι_1 and ι_2 of S^2, we obtain the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As a CW complex, $S^2 \times S^2 \cong e^4 \cup_\omega S^2 \vee S^2$, where the attaching map ω is the Whitehead product $[\iota_1, \iota_2]$.

For the general case $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$, with a chosen basis $\{\iota_1, \ldots, \iota_r\}$ of $\pi_2(X)$, a basis of $\pi_3(X)$ is given by the maps $\eta_i = \iota_i \circ \eta$ $(1 \leq i \leq r)$ and $\omega_{ij} = (\iota_i \vee \iota_j) \circ \omega = [\iota_i, \iota_j]$ $(1 \leq i < j \leq r)$. If we choose the dual basis for $H^2(X)$, $\gamma(\eta_i)$ and $\gamma(\omega_{ij})$ are represented, respectively, by the matrices H_i and Ω_{ij}, where $\left(\mathrm{H}_i\right)_{uv} = \delta_{iu}\delta_{iv}$ and $\left(\Omega_{ij}\right)_{uv} = \delta_{iu}\delta_{jv} + \delta_{iv}\delta_{ju}$.

For $f \in \pi_3(X)$, denote the bilinear form $\gamma(f)$ by Q_f. Notice that Q_f defines an element $\psi(f) \in \mathrm{Hom}(H^2(X), \mathbb{Z}/2\mathbb{Z}) \cong H_2(X; \mathbb{Z}/2\mathbb{Z})$, given by $\alpha \mapsto Q_f(\alpha, \alpha) \bmod 2$. Since $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}$, there is a natural isomorphism $\phi\colon \pi_4(\Sigma X) \cong H_2(X; \mathbb{Z}/2\mathbb{Z})$. One verifies the following, where Σf denotes the suspension of f.

Lemma 2.5 $\psi(f) = \phi(\Sigma f)$ □

Let F be a free $\mathbb{Z}$-module and let Q be a symmetric bilinear form on F. We call Q of *even type* if $Q(x, x)$ is even for all $x \in F$. Otherwise Q is called of *odd type*. The following is obvious from the definition of the map ψ and lemma 2.5.

Lemma 2.6 *The complex $e^4 \cup_f X$ has even intersection form if and only if Σf is nullhomotopic.* □

The following construction is important for later results.

Lemma 2.7 *Let $f \in \pi_3(X)$, $f \notin \ker \psi$, $Y = e^4 \cup_f X$, $a \in H_2(Y)$ an indivisible integer lift of $\psi(f) \in H_2(Y; \mathbb{Z}/2\mathbb{Z})$ and $p \in \pi_2(Y)$ its inverse image under the Hurewicz isomorphism. Let L be the cofibre of the map p. Then*

i) rank $H_2(L) = $ rank $H_2(Y) - 1$.

ii) *$L \simeq e^4 \cup_g S^2 \vee \ldots \vee S^2$ for some $g \in \pi_3(S^2 \vee \ldots \vee S^2)$.*

iii) *L has even intersection form.*

Proof: Let K be the cofibre of the map $\tilde{p}\colon S^2 \to X$ induced by p and $\tilde{q}\colon X \to K$ the projection. Then $L \simeq e^4 \cup_g K$ where $g = \hat{q} \circ f$. But $K \simeq S^2 \vee \ldots \vee S^2$, because a is indivisible. Now notice that it follows from the construction of L that the class $\psi(f) \in H_2(X; \mathbf{Z}/2\mathbf{Z})$ is mapped to zero by $\tilde{q}$. As a consequence, $g = \tilde{q}_*(f)$ is contained in $\ker\psi$, so the intersection form of L is even. □

Lemma 2.8 *If Y has odd intersection form, then there is a (non-natural) homotopy equivalence $\Sigma Y \simeq \Sigma \mathbf{C}\mathrm{P}^2 \vee (S^3 \vee \ldots \vee S^3)$.*

Proof: We use the same notations as in the proof of lemma 2.7. We already know that $\psi(f)$ is in the kernel of $\tilde{q}_*$, so it is in the image of $\tilde{p}_*$. It follows that Σf is in the image of $\tilde{p}_*\colon \pi_4(S^3) \to \pi_4(\Sigma X)$. But since $\pi_4(S^3) = \mathbf{Z}/2\mathbf{Z}$, generated by $\Sigma\eta$, and according to our assumptions $\Sigma f \neq 0$, it follows that $\Sigma f = \Sigma\tilde{p} \circ \Sigma\eta$. Now we choose a homotopy equivalence $S^3 \vee \Sigma K \simeq \Sigma X$ which restricted to S^3 is equal to $\Sigma\tilde{p}$. From this we obtain an equivalence $\Sigma\mathbf{C}\mathrm{P}^2 \vee \Sigma K \simeq \Sigma Y$. □

2.2 The Structure of the Tangent Bundle

Throughout this section let M be a connected, smooth four-dimensional manifold. Let $Pd\colon H_2(M) \to H^2(M)$ denote the Poincaré duality isomorphism. We recall some facts about the tangent bundle of M.

Lemma 2.9 *Let M be a smooth, simply-connected closed four-dimensional manifold and let $a \in H_2(M)$. Then there exists an oriented surface Σ and a smooth embedding $\Sigma \hookrightarrow M$ such that the the fundamental class of Σ in $H_2(\Sigma)$ is mapped to a.*

Proof: Let $\alpha = Pd(a)$. There is a smooth complex line bundle L_α on M with first Chern class $c_1(L) = \alpha$. The set of zeroes of a generic smooth section σ of L gives the desired 2-dimensional submanifold Σ of M. $\square$

Remark: Since Σ has codimension 2 in M, we can always assume Σ to be connected.

Now let M be an orientable smooth four-dimensional manifold with a chosen Riemannian metric, so we can assume the structure group of the tangent bundle TM to be $SO(4)$. A *spin structure* is a lift of the structure group to Spin(4), the universal cover of $SO(4)$. A *spin manifold* is a manifold with a spin structure. For simply-connected M there exists at most one spin structure. The obstruction to its existence is the second Stiefel-Whitney class $w_2(TM) \in H^2(M; \mathbf{Z}/2\mathbf{Z})$. The following lemma characterizes $w_2(TM)$ uniquely.

Lemma 2.10 *Let M be a smooth, simply-connected closed four-dimensional manifold. Then, for all $\alpha \in H^2(M; \mathbf{Z}/2\mathbf{Z})$, $(w_2(TM), \alpha) = (\alpha, \alpha)$.*

Proof: Let $\alpha \in H^2(M; \mathbf{Z}/2\mathbf{Z})$. Choose $\Sigma \subset M$ according to lemma 2.9 such that $[\Sigma]$ is an integer lift of $\alpha \cap \mu$. The bundle $TM|_\Sigma$ splits as $T\Sigma \oplus \nu_\Sigma$. Since $w_1(T\Sigma) = 0 = w_1(\nu_\Sigma)$, we obtain

$$\begin{aligned}(w_2(TM), \alpha) &= \langle w_2(TM|_\Sigma), [\Sigma] \rangle \\ &= \langle w_2(T\Sigma), [\Sigma] \rangle + \langle w_2(\nu_\Sigma), [\Sigma] \rangle\end{aligned}$$

For the surface Σ, $\langle w_2(T\Sigma), [\Sigma] \rangle = \chi(\Sigma) \bmod 2$ where χ denotes the Euler characteristic, which is even for Σ. For the normal bundle we obtain $\langle w_2(\nu_\Sigma), [\Sigma] \rangle \equiv (Pd[\Sigma], Pd[\Sigma]) \pmod 2$. $\square$

Corollary 2.11 *For a smooth simply-connected closed four-manifold, the element $\psi(f)$ in lemma 2.5 is the Poincaré dual of the second Stiefel-Whitney class of the tangent bundle.* $\square$

Corollary 2.12 *Let M have odd intersection form. Let $\Sigma \subset M$ be a smooth, connected surface representing an indivisible integer lift of the element $\psi(f)$. Then*

i) $M \setminus \Sigma \simeq S^2 \vee \ldots \vee S^2$;

ii) $TM\big|_{M \setminus \Sigma}$ *is trivial;*

iii) If L is obtained from M as in lemma 2.7, the projection $M \to L$ factors up to homotopy through M/Σ.

Proof: Let $\nu = \nu_\Sigma$ be the normal bundle of Σ in M and let ν_0 be obtained from ν by removing the zero section. Consider the cohomology long exact sequence of the pair $(M, M \setminus \Sigma)$. The positive generator of $H^2(M, M \setminus \Sigma) \cong H^2(\nu, \nu_0) \cong \mathbf{Z}$ maps to $Pd([\Sigma]) \in H^2(M)$. This implies that $H^1(M \setminus \Sigma) = 0$. Notice that $H^3(M, M \setminus \Sigma) \cong H^3(\nu, \nu_0) \cong H^1(\Sigma)$, so there is an exact sequence

$$0 \to H^2(M, M \setminus \Sigma) \to H^2(M) \to H^2(M \setminus \Sigma) \to H^1(\Sigma) \to 0.$$

Since $H^1(\Sigma)$ is free, we get an isomorphism

$$H^2(M \setminus \Sigma) \cong H^2(M) \big/ _{\langle Pd([\Sigma]) \rangle} \oplus H^1(\Sigma)$$

and since $Pd([\Sigma])$ is indivisible, we know that $H^2(M \setminus \Sigma)$ is free. Clearly $H^4(M \setminus \Sigma) = 0$, and using the exact sequence

$$0 \to H^3(M \setminus \Sigma) \to H^4(M, M \setminus \Sigma) \to H^4(M) \to 0$$

where $H^4(M, M \setminus \Sigma) \cong H^4(\nu, \nu_0) \cong H^2(\Sigma) \cong \mathbf{Z}$ we obtain $H^3(M \setminus \Sigma) = 0$. It now follows from Whitehead's theorem that $M \setminus \Sigma \simeq S^2 \vee \ldots \vee S^2$. In order to prove the second statement, it is enough to show that the second Stiefel-Whitney class w_2 of the restricted bundle is zero. But this is clear, because the integer lift $Pd([\Sigma]) \in H^2(M)$ of $w_2(TM)$ is in the image of the map $H^2(M, M \setminus \Sigma) \to H^2(M)$. The third statement is obtained by observing that the inclusion $\Sigma \hookrightarrow M$ factors up to homotopy through a map $p\colon S^2 \to M$ which satisfies the conditions of lemma 2.7, as follows from corollary 2.11. $\square$

Theorem 2.13 *For a smooth simply-connected four-dimensional manifold the following are equivalent:*

i) *M is a spin manifold;*

ii) *M has even intersection form;*

iii) *$TM\big|_{M \setminus \mathrm{pt}}$ is trivial.*

Proof: We know from lemma 2.10 that $w_2(TM)$ is zero if and only if M has even intersection form. But on the other hand, $w_2(TM)$ is the obstruction to the existence of a spin structure. To see that *(iii)* is equivalent to *(i)* and *(ii)*, observe that the bundle $TM\big|_{M \setminus \mathrm{pt}}$ is is the pull-back of the universal bundle via the classifying map $\xi\colon M \setminus \mathrm{pt} \hookrightarrow M \to BSO(4)$. But $M \setminus \mathrm{pt} \simeq S^2 \vee \ldots \vee S^2$ and the map $\xi_*\colon \pi_2(M \setminus \mathrm{pt}) \to \pi_2(BSO(4)) = \mathbb{Z}/2\mathbb{Z}$ is precisely given by evaluation with w_2. $\square$

Chapter 3

Bilinear Forms

3.1 Forms over the Integers

We recall some facts about the classification of bilinear forms. First we consider symmetric unimodular bilinear forms over the integers. All this is well known, see for example [25]. Let Q be a unimodular symmetric bilinear form on the free $\mathbb{Z}$-module F of rank r. Observe that the rational form $Q \otimes \mathbf{Q}$ over the vector space $F \otimes \mathbf{Q}$ can be diagonalized. For an orthonormal basis $\{e_1, \ldots, e_r\}$ of $F \otimes \mathbf{Q}$ let b^+ denote the number of basis vectors e_i with $Q(e_i, e_i) > 0$ and b^- the number of vectors with $Q(e_i, e_i) < 0$. Clearly $b^+ + b^- = r$. Define the *signature* $\sigma(Q)$ to be $b^+ - b^-$. One checks that the signature is well-defined. Also notice that $r \equiv \sigma \pmod 2$. A form is definite if and only if $r = \pm\sigma$.

Example 3.1 A positive definite unimodular bilinear form of even type is given by the following matrix:

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

The form has rank 8 and signature 8. A negative definite form of the same type and rank is given by the matrix $-E_8$.

Example 3.2 The intersection matrices of the examples 2.3 and 2.4 provide examples of matrices of odd type, rank 1 and signature ± 1 and of even type, rank 2 and signature 0.

Example 3.3 Two forms Q_1 on F_1 and Q_2 on F_2 give rise to a form $Q_1 \oplus Q_2$ on $F_1 \oplus F_2$ in the obvious way. From the previous examples, together with this direct sum operation, we obtain, for $m, n \in \mathbb{N}$, odd forms of rank $m + n$ and signature $m - n$ and even forms of rank $2m + 8n$ and signature $\pm 8n$.

A complete classification exists for indefinite forms.

Theorem 3.4 *The isomorphism class of an indefinite symmetric unimodular bilinear form over $\mathbb{Z}$ is uniquely characterized by its type, rank and signature.*

Proof: See [25, Chapter II, §5]. □

Let Q be a unimodular symmetric bilinear form on F. An element c of F is called *characteristic*, if, for all $x \in F$, $Q(c, x) \equiv Q(x, x) \pmod 2$. For example, it follows from lemma 2.10 that for a smooth, simply-connected closed four-manifold M, any integer lift of $w_2(TM)$ is characteristic with respect to the intersection form of M. It is not difficult to see that, for any characteristic element $c \in F$, $Q(c, c) \equiv \sigma(Q) \pmod 8$ (see [25]).

As a consequence of theorem 3.4, all indefinite odd forms are given by example 3.3. The same is true for indefinite even forms, as follows from the next lemma.

Lemma 3.5 *For a unimodular symmetric bilinear form of even type, the signature is divisible by eight.*

Proof: If the intersection form is of even type, then 0 is characteristic, so $\sigma(Q) \equiv 0 \pmod 8$. □

It follows that even forms always have even rank.

For definite forms, the analogue of theorem 3.4 is false. On the contrary, the number of isomorphism classes grows very rapidly with the rank, and their classification turns out to be very difficult. For rank 40, the number of isomorphism classes is already greater than 10^{51} ([25, Chapter II, §6]). On the other hand, S. K. Donaldson has obtained severe restrictions on the possible intersection forms of smooth manifolds (see [9, 10, 11]). In particular, if one is only interested in the intersection forms of *smooth* simply-connected 4-manifolds, the difficulties concerning definite forms do not enter.

Theorem 3.6 (Donaldson) *If X is a smooth, compact, connected, simply-connected 4-manifold with definite intersection form, then this form is diagonalizable over $\mathbb{Z}$.*

Proof: See [9, theorem A]. □

3.2 Forms over Finite Fields

Further on we also have to consider bilinear forms over finite fields. We need the following theorems, which can be found in [26].

Theorem 3.7 *Any symmetric bilinear form over $\mathbb{Z}/3\mathbb{Z}$ can be diagonalized such that there is at most one entry equal to -1 in the diagonal. In particular, unimodular bilinear forms over $\mathbb{Z}/3\mathbb{Z}$ are uniquely characterized by their rank and determinant.* □

Remark: A symmetric bilinear form can be diagonalized over any field of characteristic not equal to 2. Now observe that, in $\mathbb{Z}/3\mathbb{Z}$,

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which shows that pairs of entries of -1 in the diagonal can be cancelled. For unimodular forms of a given rank the remaining two possible equivalence classes are distinguished by the determinant, which is well-defined for bilinear forms over $\mathbb{Z}/3\mathbb{Z}$.

Corollary 3.8 *Let Q_1, Q_2 be symmetric matrices over $\mathbb{Z}/3\mathbb{Z}$. Then the following are equivalent:*

i) There is a matrix U such that $U^T Q_1 U = Q_2$ and $\det U = 1$.

ii) There is a matrix V such that $V^T Q_1 V = Q_2$ and $\det V = -1$.

Proof: For any symmetric matrix T over $\mathbb{Z}/3\mathbb{Z}$ there is a transformation matrix W of determinant -1 such that $W^T T W = T$. This is obvious if W is diagonal and then follows for general W from theorem 3.7. $\square$

Theorem 3.9 *Any symmetric bilinear form over $\mathbb{Z}/2\mathbb{Z}$ can either be diagonalized or is equivalent to a sum of copies of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and (0). In particular, a unimodular symmetric bilinear form over $\mathbb{Z}/2\mathbb{Z}$ is uniquely determined by its rank and type.*

Proof: First observe that any symmetric bilinear form over $\mathbb{Z}/2\mathbb{Z}$ is equivalent to a sum of copies of (1), $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and (0). Now one checks that, over $\mathbb{Z}/2\mathbb{Z}$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

so $(1) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is equivalent to $(1) \oplus (1) \oplus (1)$. $\square$

The following lemma will be useful to compare equivalence of forms over $\mathbb{Z}$ and over $\mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{N}$.

Lemma 3.10 *Let $m \in \mathbb{N}$ and let A be a k by k integer matrix with* $\det A \equiv \pm 1 \pmod{m}$. *Then there is an integer matrix $\tilde{A}$ such that* $\tilde{A} \equiv A \pmod{m}$ *and* $\det \tilde{A} = \pm 1$.

Proof: For the first case, $\det A \equiv 1 \pmod{m}$, see [26]. The second case, $\det A \equiv -1 \pmod{m}$, follows from the first. □

3.3 Forms over Z/4Z

Finally, we need the classification of unimodular symmetric bilinear forms over $\mathbf{Z}/4\mathbf{Z}$. We have not found this in the standard literature on the subject. There is, however, a close connection to the results on mod four valued quadratic forms on $\mathbf{Z}/2\mathbf{Z}$ vector spaces as introduced by E. H. Brown, Jr. [5] and classified by J. A. Wood [32].

Two obvious invariants of a form over $\mathbf{Z}/4\mathbf{Z}$ are its rank and its type (which depends only on the mod 2 reduction of the form).

Lemma 3.11 *Any unimodular symmetric bilinear form of odd type over $\mathbf{Z}/4\mathbf{Z}$ is diagonalizable.*

Proof: Let Q be of odd type. Since any odd element of $\mathbf{Z}/4\mathbf{Z}$ is a unit, any $\mathbf{Z}/4\mathbf{Z}$ lift of an invertible matrix over $\mathbf{Z}/2\mathbf{Z}$ is invertible. According to theorem 3.9, the mod 2 reduction of Q can be diagonalized over $\mathbf{Z}/2\mathbf{Z}$. Hence, using any $\mathbf{Z}/4\mathbf{Z}$ lift of a transformation that diagonalizes Q mod 2, we see that Q can be represented by a matrix P with odd diagonal entries and even non-diagonal entries. Notice that 2 times the k-th row of P is of the form $(0, \ldots, 0, 2, 0, \ldots, 0)$ where the entry 2 is at the k-th position. But this means that all non-diagonal entries can be made zero by simultaneous elementary row and column operations. □

This lemma puts us in the position to define a further invariant $\sigma(Q)$ of a unimodular symmetric bilinear form Q over $\mathbb{Z}/4\mathbb{Z}$. This invariant takes its values in $\mathbb{Z}/8\mathbb{Z}$, and in analogy to the integer case we call it the *signature* of Q. It is defined as follows. For a given form Q, the form $Q \oplus (1) \oplus (-1)$ can be diagonalized according to lemma 3.11. Let b^+ be the number of entries equal to 1 and b^- the number of entries equal to 3 in the diagonal. Now define

$$\sigma(Q) = b^+ - b^- \bmod 8.$$

Lemma 3.12 *$\sigma(Q)$ is well-defined.*

Proof: We have to show that the value $b^+ - b^-$ is independent of the particular choice of the diagonalization. This can be seen as follows. Let Q be a form over the free $\mathbb{Z}/4\mathbb{Z}$-module $F \otimes \mathbb{Z}/4\mathbb{Z}$, where F is a free $\mathbb{Z}$-module. Suppose $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ are two bases of $F \otimes \mathbb{Z}/4\mathbb{Z}$, such that Q is diagonal both with respect to A and B. Let σ_A be the signature of Q taken with respect to A and σ_B the signature with respect to B. Choose an integer lift of A, i. e. a basis $\hat{A} = \{\hat{a}_1, \ldots, \hat{a}_r\}$ of F such that $\hat{a}_i \bmod 4 = a_i$ for $1 \leq i \leq r$. Then define the integral bilinear form $\hat{Q}$ on F by requiring $\hat{A}$ to be an orthonormal basis and Q the mod 4 reduction of $\hat{Q}$. Then $\hat{Q}$ is unimodular and $\sigma(\hat{Q}) \bmod 8 = \sigma_A$. Now let T be the basis transformation from A to B. Lemma 3.10 ensures that we can choose an integer lift $\hat{T}$ of T, i. e. an integral basis transformation such that $\hat{T}(v) \bmod 4 = T(v \bmod 4)$ for all $v \in F$. Let $\hat{b}_i = \hat{T}(\hat{a}_i)$. It follows that $\{\hat{b}_i \mid 1 \leq i \leq r\}$ is an integral lift of B. Since Q is diagonal with respect to B, we can write $\hat{Q}(\hat{b}_i, \hat{b}_j)$ as $4 \cdot Q_{ij}$ for $i \neq j$, and $\hat{Q}(\hat{b}_i, \hat{b}_i)$ as $\varepsilon_i + 4\gamma_i$ where $\varepsilon_i \in \{-1, 1\}$ and $Q_{ij}, \gamma_i \in \mathbb{Z}$. Now let $\hat{b} = \hat{b}_1 + \cdots + \hat{b}_r$. Notice that $\hat{b}$ is characteristic, so $\hat{Q}(\hat{b}, \hat{b}) \equiv \sigma(\hat{Q}) \pmod 8$.

In order to complete the argument, we show that $\tilde{Q}(\tilde{b}, \tilde{b}) \equiv \sigma_B \pmod 8$.

$$\begin{aligned} \tilde{Q}(\tilde{b}, \tilde{b}) &= \sum_i \tilde{Q}(\tilde{b}_i, \tilde{b}_i) + 2\sum_{i<j} \tilde{Q}(\tilde{b}_i, \tilde{b}_j) \\ &= \sum_i \varepsilon_i + 4\sum_i \gamma_i + 8\sum_{i<j} Q_{ij}. \end{aligned}$$

Notice that $\sum_i \varepsilon_i = \sigma_B$, so all we need to show is that $\sum_i \gamma_i$ is even.

$$\begin{aligned} \det \tilde{Q} &= \sum_{\sigma\in\Sigma_r} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{r} \tilde{Q}(\hat{b}_i, \hat{b}_{\sigma(i)}) \\ &= \prod_{i=1}^{r}(\varepsilon_i + 4\gamma_i) + \sum_{\substack{\sigma\in\Sigma_r \\ \sigma\neq\mathrm{id}}} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{r} \tilde{Q}(\tilde{b}_i, \tilde{b}_{\sigma(i)}) \\ &= \prod_{i=1}^{r} \varepsilon_i + 4\sum_j \left(\gamma_j \cdot \prod_{i\neq j} \varepsilon_i\right) + 16K \end{aligned}$$

for some $K \in \mathbb{Z}$. Now $\det \tilde{Q} \in \{-1, 1\}$, so $\det \tilde{Q} = \prod_{i=1}^{r} \varepsilon_i$ and

$$\sum_j \left(\gamma_j \cdot \prod_{i\neq j} \varepsilon_i\right) = -4K.$$

Reducing both sides of this equation mod 2, we finally obtain

$$\sum_j \gamma_j \equiv 0 \pmod 2,$$

which is all we had to show. $\square$

Lemma 3.13 *The signature of unimodular symmetric bilinear forms over $\mathbb{Z}/4\mathbb{Z}$ has the following properties.*

i) $\sigma(P \oplus Q) = \sigma(P) + \sigma(Q)$

ii) $\mathrm{rank}(Q) \equiv \sigma(Q) \pmod 2$

iii) *If Q is the* mod 4 *reduction of an integral form $\tilde{Q}$, then $\sigma(Q)$ is the* mod 8 *reduction of $\sigma(\tilde{Q})$.*

Proof: The first statement is clear if both forms are odd. In general, notice that for any form Q, $\sigma(Q) = \sigma(Q \oplus (1) \oplus (-1))$, so

$$\begin{aligned}
\sigma(P \oplus Q) &= \sigma(P \oplus Q \oplus (1) \oplus (-1) \oplus (1) \oplus (-1)) \\
&= \sigma(P \oplus (1) \oplus (-1)) + \sigma(Q \oplus (1) \oplus (-1)) \\
&= \sigma(P) + \sigma(Q)
\end{aligned}$$

In the same way, the second statement is clearly true for odd forms, so

$$\begin{aligned}
\operatorname{rank}(Q) &\equiv \operatorname{rank}(Q \oplus (1) \oplus (-1)) \pmod 2 \\
&\equiv \sigma(Q \oplus (1) \oplus (-1)) \pmod 2 \\
&= \sigma(Q).
\end{aligned}$$

In order to show the third statement, notice that $Q\oplus(1)\oplus(-1)$ is diagonalizable over the integers, so the third statement is clearly true for $Q\oplus(1)\oplus(-1)$. The general case follows from the additivity of the signature and the fact that $(1) \oplus (-1)$ has signature zero both over the integers and over $\mathbf{Z}/4\mathbf{Z}$. □

Two important examples of unimodular symmetric bilinear forms over $\mathbf{Z}/4\mathbf{Z}$ are given by $\mathcal{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{K} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Lemma 3.14 *Any unimodular symmetric bilinear form of even type over $\mathbf{Z}/4\mathbf{Z}$ is isomorphic to an orthogonal sum of k copies of $\mathcal{H}$ ($k \geq 0$) and at most one copy of $\mathcal{K}$.*

Proof: As a consequence of theorem 3.9, we can for any unimodular even form over $\mathbf{Z}/4\mathbf{Z}$ find a basis such that the matrix has odd entries precisely at the positions $(2k-1, 2k)$ and $(2k, 2k-1)$ for $1 \leq k \leq r$. As in the proof of lemma 3.11, considering the double of each row, we see that for each k we get a vector of the form $(0, \dots, 0, 2, 0, \dots, 0)$, so again we can eliminate all even non-diagonal entries by elementary simultaneous row and column operations. Similarly, by multiplying a row and corresponding column by 3, we can change a non-diagonal entry from 3 to 1. Hence we remain with an orthogonal sum of copies of $\mathcal{H}$ and $\mathcal{K}$. But the calculation

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 2 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

shows that $\mathcal{K} \oplus \mathcal{K}$ is equivalent to $\mathcal{H} \oplus \mathcal{H}$, so pairs of summands of $\mathcal{K}$ can be replaced such that we end up with at most one summand of this kind. $\square$

We are now in the position to give a classification of unimodular symmetric bilinear forms over $\mathbb{Z}/4\mathbb{Z}$.

Theorem 3.15 *If two unimodular symmetric bilinear forms over $\mathbb{Z}/4\mathbb{Z}$ have the same rank, type and signature, they are isomorphic.*

Proof: We start with even forms. Lemma 3.14 ensures that there are at most two isomorphism classes for any given rank. It follows from the integer case that $\sigma(\mathcal{H}) = 0$. The calculation

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

shows that $\mathcal{K} \oplus (1)$ is isomorphic to $(-1) \oplus (-1) \oplus (-1)$ so $\sigma(\mathcal{K}) = 4$. Hence there are precisely two distinct isomorphism classes for given even rank, and they are distinguished by the signature. The case of odd forms reduces to showing that $4 \cdot (-1)$ and $4 \cdot (1)$ are equivalent. The form $\mathcal{K} \oplus (-1)$ is odd of rank 3 and signature 3, so it must be equivalent to $(1) \oplus (1) \oplus (1)$. Hence $\mathcal{K} \oplus (-1) \oplus (1)$ is isomorphic to $4 \cdot (1)$, so $4 \cdot (-1)$ is isomorphic to $4 \cdot (1)$. $\square$

Notice that the determinant is well-defined for forms over $\mathbb{Z}/4\mathbb{Z}$. It is related to the rank and signature in the following way.

Lemma 3.16 *Let Q be a unimodular symmetric bilinear form over $\mathbb{Z}/4\mathbb{Z}$. Then*

$$\sigma(Q) \equiv \operatorname{rank} Q + \det Q - 1 \pmod 4.$$

Proof: Consider the form $P = Q \oplus (1) \oplus (-1)$. Choose a diagonalization of P and let b^+ and b^- be the number of entries equal to 1 and 3 respectively in the corresponding matrix. Then, calculating in $\mathbb{Z}/4\mathbb{Z}$, we get

$$\begin{aligned}
\operatorname{rank} Q + \det Q - 1 &= \operatorname{rank} P - 2 - \det P - 1 \\
&= b^+ + b^- - 2 - (1+2)^{b^-} - 1 \\
&= b^+ + b^- + 1 - (1 + 2b^-) \\
&= \sigma(Q)
\end{aligned}$$

□

Remark: Let F be a $\mathbb{Z}/2\mathbb{Z}$ vector space of finite rank. A *mod four valued quadratic form Q* on F is a map $Q\colon F \to \mathbb{Z}/4\mathbb{Z}$ such that

$$Q(u+v) = Q(u) + Q(v) + j \circ B(u,v)$$

where B is a unimodular bilinear form on F and $j\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is the non-trivial embedding. The σ-invariant of Brown assigns to a mod four valued quadratic form Q the value $\sigma(Q) \in \mathbb{Z}/8\mathbb{Z}$ (see [5]) and J. A. Wood shows in [32] that the rank of Q, the type of the associated bilinear form B and $\sigma(Q)$ form a complete set of invariants for mod four valued quadratic forms.

Now suppose P is a symmetric unimodular bilinear form on the free $\mathbb{Z}/4\mathbb{Z}$ module V. Define a map $\tilde{P}\colon V \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ by $\tilde{P}(v) = P(\tilde{v}, \tilde{v})$ for some lift $\tilde{v} \in V$ of $v \in V \otimes \mathbb{Z}/2\mathbb{Z}$. One readily checks that this definition does not depend on the choice of the lift and that $\tilde{P}$ is a mod four valued quadratic form on $V \otimes \mathbb{Z}/2\mathbb{Z}$ with associated bilinear form $P \otimes \mathbb{Z}/2\mathbb{Z}$. One can show that with these definitions, $\sigma(P) = \sigma(\bar{P})$. This gives on one hand an alternative proof for lemma 3.12. On the other hand, together with Wood's classification it shows that a mod four valued quadratic form comes from a bilinear form over $\mathbb{Z}/4\mathbb{Z}$ if and only if a bilinear form exists for the given values of rank, σ and type.

Chapter 4

Mapping Spaces

4.1 The Classifying Space of the Gauge Group

For the spaces X and Y let $\mathcal{C}(X, Y)$ denote the space of continuous maps from X to Y with the compact-open topology. If X and Y have basepoints, let $\mathrm{Maps}(X, Y) \subset \mathcal{C}(X, Y)$ be the subspace of basepoint-preserving maps.

Let M be a connected, smooth closed manifold, G a compact Lie group and P a (smooth) principal G-bundle over M. The bundle P is isomorphic to the pull-back of the universal bundle $G \hookrightarrow EG \to BG$ via a suitable map $f: M \to BG$ and the isomorphism class of P and the homotopy class of f determine each other uniquely. The group $\mathcal{G}_P$ of gauge transformations of P acts freely on the contractible space $\mathcal{C}_G(P, EG)$ of G-equivariant maps from P to EG. The quotient space $\mathcal{C}_P(M, BG)$ of maps homotopic to f is a classifying space $B\mathcal{G}_P$ (see [17]). Let $* \in M$ be the basepoint and let $\mathcal{G}_P^* \subset \mathcal{G}_P$ be the subgroup leaving the fibre over $*$ fixed. Then in the same way as above, $\mathcal{G}_P^*$ acts freely on the contractible space $\mathrm{Maps}_G(P, EG)$ where a basepoint of P is chosen in the fibre over $* \in M$, and $\mathrm{Maps}_P(M, BG)$ is a $B\mathcal{G}_P^*$.

Let $\mathcal{A}_P$ be the space of connections on P. The group $\mathcal{G}_P$ acts on $\mathcal{A}_P$ via pull-backs and the subgroup $\mathcal{G}_P^*$ acts freely. Denote the quotient $\mathcal{A}_P/\mathcal{G}_P^*$ by

$\bar{\mathcal{B}}_P$. The space $\mathcal{A}_P$, being an affine space, is contractible, so $\bar{\mathcal{B}}_P$ is also a classifying space of $\mathcal{G}_P^\bullet$. In fact, a weak homotopy equivalence

$$\delta\colon \bar{\mathcal{B}}_P \to \mathrm{Maps}_P(M, BG)$$

is constructed in the following way: Consider the principal G-bundle

$$G \hookrightarrow \mathcal{A}_P \times P \to \mathcal{A}_P \times M.$$

The group $\mathcal{G}_P^\bullet$ acts freely on $\mathcal{A}_P$, trivially on M and by definition on P, and by forming quotients we get the principal G-bundle

$$G \hookrightarrow \mathcal{A}_P \times_{\mathcal{G}_P^\bullet} P \longrightarrow \bar{\mathcal{B}}_P \times M.$$

This bundle determines the homotopy class of its classifying map

$$\gamma\colon \bar{\mathcal{B}}_P \times M \to BG$$

and the map δ can be defined by $\delta(b)(m) = \gamma(b, m)$ for $b \in \bar{\mathcal{B}}_P$, $m \in M$. For details, see [12].

Now suppose further that M is simply-connected and four-dimensional with a chosen Riemannian metric on the tangent bundle and let $G = SU(2)$. This is the setting considered in four-dimensional Yang-Mills theory, where one considers the moduli space $\mathcal{M}_P$ of self-dual instantons, i.e. gauge equivalence classes of connections satisfying a certain set of partial differential equations, the so-called self-dual Yang-Mills equations. For a generic metric, the spaces $\mathcal{M}_P \subset \bar{\mathcal{B}}_P$ are finite dimensional manifolds and in some sense finite dimensional approximations of $\bar{\mathcal{B}}_P$. Of particular interest is the induced map in homology

$$\iota_*\colon H_*(\mathcal{M}_P) \hookrightarrow H_*(\bar{\mathcal{B}}_P)$$

(see [3, 4] and chapter 6). S. K. Donaldson uses this construction to obtain obstructions for the existence of smooth structures on a topological manifold (see [9, 10]) and invariants of smooth four-manifolds (see [11]). The definition of these so-called Donaldson polynomial invariants uses, at least formally, the above inclusion ι^* in cohomology.

Our aim is to study the homology and the homotopy types of the spaces $\bar{\mathcal{B}}_P$. The above discussion shows that there is a weak equivalence

$$\bar{\mathcal{B}}_P \cong \mathrm{Maps}_P(M, BSU(2)).$$

The homotopy type of the latter space is independent of the bundle P (see theorem 4.1 below) so it depends only on the homotopy type of M, which as seen in chapter 2 is uniquely determined by the intersection form of M. One aim of this chapter is to express the homology of $\bar{\mathcal{B}}_P$ in terms of the intersection form. On the other hand, not the complete information of the intersection form is contained in the homotopy type of $\bar{\mathcal{B}}_P$, and we also pursue the question, what information about the intersection form can be retrieved from the homotopy type of $\bar{\mathcal{B}}_P$.

4.2 The Homotopy Groups of the Mapping Space

As in chapter 2, consider spaces of the form $\mathrm{Maps}(Y, BSU(2))$, where Y is a complex homotopy equivalent to a bouquet of 2-spheres with a single 4-cell attached. Recall that $BSU(2) \simeq \mathbb{H}P^\infty$ and $H^*(BSU(2)) \cong \mathbb{Z}[c]$, the polynomial algebra generated by the universal second Chern class c in dimension 4. Let $f \in \mathrm{Maps}(Y, BSU(2))$. Define the *degree* of f as

$$\deg(f) = \langle f^*(c), \mu \rangle \in \mathbb{Z},$$

where $\mu \in H_4(Y)$ is the chosen orientation class. Since the 5-skeleton of $\mathbb{H}P^\infty$ is $\mathbb{H}P^1 \cong S^4$, one sees that two maps from Y to $BSU(2)$ are homotopic if and only if they are of the same degree, i. e. $SU(2)$-bundles over Y are up to isomorphism determined by their second Chern class. For $k \in \mathbb{Z}$ let $\mathrm{Maps}_k(Y, BSU(2))$ denote the subspace consisting of maps of degree k.

Theorem 4.1 *There is a homotopy equivalence*

$$\mathrm{Maps}(Y, BSU(2)) \simeq \mathbb{Z} \times \mathrm{Maps}_0(Y, BSU(2))$$

Proof: For $s \in \mathbb{Z}$ we construct a homotopy equivalence

$$\phi_s\colon \mathrm{Maps}_k(Y, BSU(2)) \to \mathrm{Maps}_{k+s}(Y, BSU(2))$$

in the following way: Fix a 'pinching map' $p\colon Y \to Y \vee S^4$ and, for $s \in \mathbb{Z}$, a standard map $\chi_s\colon S^4 \to BSU(2)$ of degree s. Define ϕ_s by the formula $\phi_s(f) = (f \vee \chi_s) \circ p$. The map ϕ_s is continuous, raises the degree by s and $\phi_{(-s)}$ is its homotopy inverse. □

Remark: The above proof uses in an essential way the fact that the maps considered preserve basepoints. The analogous statement for $\mathcal{C}(Y, BSU(2))$ is false (see [19]).

As in chapter 2, let $X \simeq S^2 \vee \ldots \vee S^2$ and $Y \simeq e^4 \cup_f X$. Applying the functor $\mathrm{Maps}(\,\cdot\,, BSU(2))$ to the cofibration sequence

$$S^3 \xrightarrow{f} X \longrightarrow Y \longrightarrow S^4 \longrightarrow \cdots$$

we obtain a fibration sequence

$$\cdots \to \Omega^4 BSU(2) \to \mathrm{Maps}(Y, BSU(2)) \to \mathrm{Maps}(X, BSU(2)) \xrightarrow{f^\sharp} \Omega^3 BSU(2)$$

where we have written $\Omega^n BSU(2)$ for $\mathrm{Maps}(S^n, BSU(2))$. Since $SU(2) \simeq \Omega BSU(2)$ and $SU(2) \cong S^3$, we know that $\pi_1(\Omega^3 BSU(2)) \cong \pi_1(\Omega^2 S^3) \cong \pi_3(S^3) \cong \mathbb{Z}$, and since $\mathrm{Maps}(X, BSU(2))$ is simply connected, the boundary

$$\partial\colon \pi_1(\Omega^3 BSU(2)) \longrightarrow \pi_0(\mathrm{Maps}(Y, BSU(2))$$

in the homotopy long exact sequence is a bijection. Hence if we replace $\Omega^3 BSU(2)$ by its universal cover $\Omega^2\mathcal{F}$, where $\mathcal{F}$ is the 3-connected cover $S^3_{<3>}$ of S^3, we obtain the fibration sequence

$$(4.1)\quad \cdots \to \Omega^3_0 S^3 \to \mathrm{Maps}_0(Y, BSU(2)) \to \mathrm{Maps}(X, BSU(2)) \xrightarrow{f^\sharp} \Omega^2\mathcal{F}$$

which is the basis for our further calculations. In the following we denote by $f^\sharp$ the map from $\mathrm{Maps}(X, BSU(2))$ to $\mathrm{Maps}(S^3, BSU(2))$ given by composition on the right with f as well as its unique lift to the universal cover $\Omega^2\mathcal{F}$. The appropriate meaning will always be clear from the context.

Lemma 4.2 $\Omega^2 S^3 \simeq \Omega^2\mathcal{F} \times S^1$.

Proof: Let $\gamma\colon S^1 \to \Omega^2 S^3$ be a generator of π_1. Using the loop space structure on $\Omega^2 S^3$, we can define a homotopy equivalence $S^1 \times \Omega^2\mathcal{F} \to \Omega^2 S^3$. □

Lemma 4.3 *For all $n \in \mathbb{N}$, there is a natural isomorphism*

$$\pi_n\Big(\mathrm{Maps}(X, BSU(2))\Big) \cong H^2(X) \otimes \pi_{n+1}(S^3)$$

□

Remark: Notice that the adjoint of the above isomorphism is induced by the composition

$$\begin{aligned}\pi_2(X) \times \pi_n\Big(\mathrm{Maps}(X, BSU(2))\Big) &\xrightarrow{\wedge} \pi_{n+2}\Big(X \wedge \mathrm{Maps}(X, BSU(2))\Big)\\ &\xrightarrow{(\mathrm{eval})_*} \pi_{n+2}(BSU(2)).\end{aligned}$$

Theorem 4.4 *Suppose Y has even intersection form. Then, for $n \geq 1$, there is an isomorphism*

$$\pi_n\Big(\mathrm{Maps}(Y, BSU(2))\Big) \cong \pi_{n+3}(S^3) \oplus H^2(Y) \otimes \pi_{n+1}(S^3).$$

Proof: Notice that, for $n \geq 1$,

$$\pi_n\Big(\mathrm{Maps}(Y, BSU(2))\Big) \cong \pi_{n-1}\Big(\mathrm{Maps}(\Sigma Y, BSU(2))\Big).$$

Since the intersection form of Y is even, it follows from lemma 2.6 that the suspension of f is null-homotopic. Hence the inclusion $\Sigma X \to \Sigma Y$ has a homotopy left inverse, which, for $n \geq 1$, induces a splitting

$$\pi_n\Big(\mathrm{Maps}(X, BSU(2))\Big) \to \pi_n\Big(\mathrm{Maps}(Y, BSU(2))\Big)$$

and the homotopy long exact sequence falls into short split exact sequences

$$0 \to \pi_{n+1}\Big(\Omega^2 S^3\Big) \to \pi_n\Big(\mathrm{Maps}(Y, BSU(2))\Big) \rightleftarrows \pi_n\Big(\mathrm{Maps}(X, BSU(2))\Big) \to 0.$$

Now the result follows from lemma 4.3. $\square$

In order to obtain results for complexes with odd intersection form, consider the map

$$(f^\sharp)_*\colon \pi_n\Big(\mathrm{Maps}(X, BSU(2))\Big) \longrightarrow \pi_n\Big(\Omega^2\mathcal{F}\Big)$$

in homotopy. In view of lemma 4.3, we can describe $f^\sharp$ in the following way.

Theorem 4.5 *For $n \geq 1$, the map $f^\sharp_*\colon H^2(X) \otimes \pi_{n+1}(S^3) \to \pi_{n+2}(S^3)$ is given by the formula*

$$\alpha \otimes y \longmapsto Q_f(\alpha, \alpha) \cdot \eta_\sharp(y),$$

for $\alpha \in H^2(X)$, $y \in \pi_{n+1}(S^3)$, where Q_f is the intersection form corresponding to $f \in \pi_3(X)$ and $\eta_\sharp(y) = y \circ \Sigma^{n-1}\eta$.

Proof: Notice that we only have to check the case $n \geq 2$, and since $\Sigma\eta$ has order two, we only have to consider odd intersection forms. As in lemma 2.7, let $p \in \pi_2(X)$ be the inverse image under the Hurewicz isomorphism of an indivisible integer lift ξ of $\psi(f) \in H_2(X;\mathbb{Z}/2\mathbb{Z})$. We have seen in the proof of lemma 2.8 that $\Sigma f \simeq \Sigma p \circ \Sigma \eta$. Hence the map

$$(p \circ \eta)^\sharp_n : \pi_n\Big(\mathrm{Maps}(X, BSU(2))\Big) \longrightarrow \pi_n(\Omega^2\mathcal{F})$$

induced by $p \circ \eta \in \pi_3(X)$ agrees with $f^\sharp_n$ for $n \geq 2$. Thus we obtain the following factorization for $f^\sharp$

$$H^2(X) \otimes \pi_{n+1}(S^3) \xrightarrow{p^*\otimes \mathrm{id}} H^2(S^2) \otimes \pi_{n+1}(S^3) \xrightarrow{\eta_\sharp} \pi_{n+2}(S^3).$$

All that remains to be shown is that, for $\alpha \in H^2(X)$, $\langle p^*(\alpha), [S^2]\rangle \equiv Q_f(\alpha,\alpha)$ (mod 2), where $[S^2] \in H_2(S^2)$ is the standard generator. But, using lemma 2.5, we get

$$Q_f(\alpha,\alpha) \bmod 2 = \langle \alpha\,,\, \psi(f)\rangle = \langle \alpha\,,\, \xi\rangle \bmod 2 = \Big\langle p^*(\alpha)\,,\,[S^2]\Big\rangle \bmod 2$$

□

As a first consequence of this result, we see that homotopy groups distinguish between odd and even intersection forms.

Corollary 4.6

$$\pi_1\Big(\mathrm{Maps}(Y, BSU(2))\Big) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \textit{if } Y \textit{ has even intersection form} \\ 0 & \textit{if } Y \textit{ has odd intersection form} \end{cases}$$

□

It follows from theorem 4.4 that the homotopy groups of $\mathrm{Maps}(Y, BSU(2))$ for a complex Y with even intersection form only depend of the rank of $H^2(Y)$. The following theorem states that the same is true for complexes with odd intersection form.

Theorem 4.7 *Let Y_1 and Y_2 be complexes with odd intersection forms of the same rank. Then, for all $n \in \mathbb{N}$,*

$$\pi_n\Big(\mathrm{Maps}(Y_1, BSU(2))\Big) \cong \pi_n\Big(\mathrm{Maps}(Y_2, BSU(2))\Big).$$

Proof: We know from corollary 4.6 that both mapping spaces are simply-connected. As a consequence of lemma 2.8 there is a homotopy equivalence $\Sigma Y_1 \simeq \Sigma Y_2$, which gives the isomorphisms for $n \geq 2$. $\square$

Remark: In low dimensions, where the homotopy groups of S^3 and the effect of the map $\eta_\sharp$ are well known, we can apply theorem 4.5 to calculate the homotopy groups of the mapping space. Based on the information in [30] we obtain the following values:

n	0	1	2	3	4	5	6	7	8
$\pi_n\Big(\mathrm{Maps}(\mathbb{C}P^2, BSU(2))\Big)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	0	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/30\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$

In arbitrary dimensions we obtain a short exact sequence

$$0 \to coker\, f^\sharp_{n+1} \to \pi_n\Big(\mathrm{Maps}(Y, BSU(2))\Big) \to ker\, f^\sharp_n \to 0$$

but we do not know whether it always splits. For the odd primary components this is always the case:

Theorem 4.8 *Let p be an odd prime and let $\pi_n(\,\cdot\,;p)$ denote the p-primary component of π_n. Then*

$$\pi_n\Big(\mathrm{Maps}(Y, BSU(2)); p\Big) \cong \pi_{n+3}(S^3;p) \oplus H^2(Y) \otimes \pi_{n+1}(S^3;p)$$

Proof: For complexes Y with even intersection form this is a consequence of theorem 4.4. For odd intersection forms it follows from the properties of $\eta_\sharp$ that $f^\sharp$ is zero on p-primary components. To establish the splitting of the above sequence it is in view of lemma 2.8 enough to consider the case $Y = \mathbb{C}P^2$. Let $K = e^4 \cup_{2\eta} S^2$. There is a commutative diagram of cofibrations

$$\begin{tikzcd}
S^3 \arrow[r, "2\eta"] \arrow[d, "\times 2"] & S^2 \arrow[r] \arrow[d, equal] & K \arrow[d, dash] \\
S^3 \arrow[r, "\eta"] & S^2 \arrow[r] & \mathbb{C}P^2
\end{tikzcd}$$

where $\times 2$ denotes a degree 2 map. Apply the functor $\pi_*\big(\mathrm{Maps}(\,\cdot\,, BSU(2))\big)$ to the diagram to get a map of two long exact homotopy sequences. The map $\times 2$ induces multiplication by 2 on $\pi_*\big(\mathrm{Maps}(S^3, BSU(2))\big)$ which is an isomorphism on p-primary parts. Thus the five-lemma gives an isomorphism

$$\pi_*\big(\mathrm{Maps}(\mathbb{C}P^2, BSU(2)); p\big) \cong \pi_*\big(\mathrm{Maps}(K, BSU(2)); p\big)$$

and the result follows since K has even intersection form. □

4.3 The Homology of Maps(X,BSU(2))

In order to proceed further in our calculations, we need the following results about the homology of loop spaces. As a general reference, see [7]. Consider the space $\Omega T = \mathrm{Maps}(S^1, T)$ of (based) loops on a space T. The loop structure induces a multiplication $m : \Omega T \times \Omega T \to \Omega T$ and in homology the so-called *Pontryagin product*

$$m_* : H_*(\Omega T) \otimes H_*(\Omega T) \longrightarrow H_*(\Omega T).$$

Now let R be a ring such that $H_*(\Omega T; R)$ is a flat R-module. The diagonal map, together with the Künneth formula, induces a comultiplication on $H_*(\Omega T; R)$. Both structures together make $H_*(\Omega T; R)$ a graded Hopf algebra. Similarly, if $H^*(\Omega T; R)$ is a flat R-module of finite type, this module gets the structure of a graded Hopf algebra via the cup product and coproduct m^*. For $T = S^{n+1}$, the following result is well-known. See for example [31].

Theorem 4.9 *There is an isomorphism of Hopf algebras*

$$H_*(\Omega S^{n+1}) \cong \mathbb{Z}[b],$$

where $\mathbb{Z}[b]$ denotes the polynomial Hopf algebra in one primitive generator $b \in H_n(\Omega S^{n+1})$. □

Let F be a finitely generated free $\mathbb{Z}$-module and let $F^* = \mathrm{Hom}(F, \mathbb{Z})$ be its dual. We can give the symmetric algebra $\mathcal{S}^*(F)$ a Hopf algebra structure by requiring elements of $\mathcal{S}^1(F)$ to be primitive. Let $\Gamma_*(F^*) = \mathrm{Hom}(\mathcal{S}^*(F), \mathbb{Z})$ be the dual Hopf algebra, the so-called *divided power algebra* on F^*. As always, let $X \simeq S^2 \vee \ldots \vee S^2$.

Theorem 4.10 *There is an H-space structure on* $\mathrm{Maps}(X, BSU(2))$ *which is, up to homotopy, natural with respect to maps $X_1 \to X_2$ of spaces homotopy equivalent to bouquets of 2-spheres, and there are natural isomorphisms of Hopf algebras*

$$\begin{aligned} H_*(\mathrm{Maps}(X, BSU(2))) &\cong \mathcal{S}^*(H^2(X)) \\ H^*(\mathrm{Maps}(X, BSU(2))) &\cong \Gamma_*(H_2(X)). \end{aligned}$$

Proof: Choose a homotopy equivalence $\phi\colon S^2 \vee \ldots \vee S^2 \to X$. The standard co-H-space structure on $S^2 \vee \ldots \vee S^2$ induces via ϕ a co-H-space structure on X. Since any homotopy equivalence $S^2 \vee \ldots \vee S^2 \to S^2 \vee \ldots \vee S^2$ desuspends, the induced structure on X is unique up to homotopy. Now, applying the functor $\mathrm{Maps}(\,\cdot\,, BSU(2))$ we get a homotopy equivalence of H-spaces

$$\mathrm{Maps}(X, BSU(2)) \xrightarrow{\simeq} \Omega S^3 \times \cdots \times \Omega S^3$$

and it follows from theorem 4.9 that there is an isomorphism of Hopf algebras

$$H_*(\mathrm{Maps}(X, BSU(2)) \cong \mathcal{S}^*\big(H_2(\mathrm{Maps}(X, BSU(2)))\big).$$

But $H_2(\mathrm{Maps}(X, BSU(2)))$ is naturally isomorphic to $H^2(X)$, as follows from lemma 4.3 together with the Hurewicz theorem, which establishes the first statement. The second statement follows by dualizing the first. □

4.4 The Homotopy Type

Let $Y \simeq e^4 \cup_f S^2 \vee \ldots \vee S^2$ and $Z \simeq e^4 \cup_g S^2 \vee \ldots \vee S^2$. The aim of this section is to prove the following theorem.

Theorem 4.11 *Suppose the intersection forms of Y and Z are equivalent over $\mathbb{Z}/12\mathbb{Z}$. Then the spaces* $\mathrm{Maps}(Y, BSU(2))$ *and* $\mathrm{Maps}(Z, BSU(2))$ *have the same homotopy type.*

This theorem has the following consequences.

Corollary 4.12 *Let M and N be closed simply-connected four-manifolds with intersection forms Q_M and Q_N respectively and suppose that Q_M and Q_N are of the same rank and type. If $\sigma(Q_M) + \sigma(Q_N)$ or $\sigma(Q_M) - \sigma(Q_N)$ is divisible by eight, then there is a homotopy equivalence*

$$\mathrm{Maps}(M, BSU(2)) \simeq \mathrm{Maps}(N, BSU(2)).$$

Proof: Since reversing the orientation of M changes the sign of the signature of Q_M, we can without loss of generality assume that $\sigma(Q_M) \equiv \sigma(Q_N) \pmod 8$. Now lemma 3.16 implies that $\det Q_M \equiv \det Q_N \pmod 4$ and unimodularity implies that $\det Q_M = \det Q_N$. Hence Q_M and Q_N have isomorphic mod 3 reductions according to theorem 3.7 and isomorphic mod 4 reductions according to theorem 3.15. It follows from the Chinese remainder theorem that the mod 12 reductions of Q_M and Q_N are isomorphic. Now apply theorem 4.11. □

Corollary 4.13 *Let M and N be closed simply-connected four-manifolds with even intersection forms. Then* $\mathrm{Maps}(M, BSU(2))$ *and* $\mathrm{Maps}(N, BSU(2))$ *are homotopy equivalent if and only if $H^2(M)$ and $H^2(N)$ have the same rank.*

Proof: According to lemma 3.5, the signatures of the intersection forms of M and N are both divisible by eight, so the second statement implies the first by corollary 4.12. To show the converse, recall from theorem 4.4 that the rank of the intersection form is equal to the rank of the free part of the second homotopy group of the mapping space. □

Let $X \simeq S^2 \vee \ldots \vee S^2$. An element f of $\pi_3(X)$ defines the homotopy class of $f^\sharp$ in $[\mathrm{Maps}(X, BSU(2)), \Omega^2\mathcal{F}]$. This set of homotopy classes is an abelian group, since $\Omega^2\mathcal{F}$ is a loop space. The map

$$\sharp : \pi_3(X) \longrightarrow [\mathrm{Maps}(X, BSU(2)), \Omega^2\mathcal{F}]$$

is a group homomorphism with kernel $12\pi_3(X)$. This was first shown by G. Masbaum (see [20]). We give a proof which stays completely within the topological category. In order to prove theorem 4.11, we only require the inclusion $12\pi_3(X) \subseteq \mathit{ker}\,\sharp$, which we establish first. Recall that for any based space W there is a 'generalized Whitehead product' map

$$h_W : \Sigma W \wedge W \longrightarrow \Sigma W \vee \Sigma W$$

with cofibre $\Sigma W \times \Sigma W$, given in coordinates by

$$t \wedge u \wedge w \longmapsto \begin{cases} 2t \wedge u & t \leq \frac{1}{2} \\ (2t-1) \wedge v & t \geq \frac{1}{2} \end{cases} .$$

This construction is natural, i. e. for spaces W, Z and maps $f, g \colon W \to Z$ there is a commutative diagram

$$(4.2) \qquad \begin{tikzcd} \Sigma W \wedge W \arrow[r, "\Sigma f \wedge g"] \arrow[d, "h_W"] & \Sigma Z \wedge Z \arrow[d, "h_Z"] \\ \Sigma W \vee \Sigma W \arrow[r, "\Sigma f \vee \Sigma g"] & \Sigma Z \vee \Sigma Z \end{tikzcd}$$

For $W = S^{n-1}$, $h_W : S^{2n-1} \to S^n \vee S^n$ is the ordinary Whitehead product $[\iota_1, \iota_2]$. Now define a map

$$\phi : \Omega S^3 \times \Omega S^3 \longrightarrow \Omega^3 S^7$$

by the formula $(f, g) \mapsto \Sigma f \wedge g$. Let $E\colon S^3 \to \Omega S^4$ be the natural inclusion, $i\colon S^4 \to BSU(2)$ the standard inclusion of $\mathbf{HP}^1$ in $\mathbf{HP}^\infty$, $\omega = [\iota_1, \iota_2] \in \pi_3(S^2 \vee S^2)$, $\iota \in \pi_4(S^4)$ the standard generator and $[\iota, \iota] \in \pi_7(S^4)$ its Whitehead square. The following observation is due to G. Masbaum [19].

Lemma 4.14 *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega S^3 \times \Omega S^3 \arrow[r, "\phi"] \arrow[d, "\Omega E \times \Omega E"] & \Omega^3 S^7 \arrow[d, "\Omega^3[\iota,\iota]"] \\
\Omega^2 S^4 \times \Omega^2 S^4 \arrow[r, "\circ\omega"] \arrow[d, "\Omega^2 i \times \Omega^2 i"] & \Omega^3 S^4 \arrow[d, "\Omega^3 i"] \\
\Omega^2 BSU(2) \times \Omega^2 BSU(2) \arrow[r, "\circ\omega"] & \Omega^3 BSU(2)
\end{tikzcd}$$

Proof: The commutativity of the bottom square is obvious, because the horizontal maps compose on the right and the vertical maps on the left. The commutativity of the top square follows from the commutative diagram (4.2) above with $W = S^1$ and $Z = S^3$. □

Remark: Notice that the composition $\Omega i \circ E \colon S^3 \to \Omega BSU(2)$ is the standard homotopy equivalence. It follows that the composition of the arrows on the left hand side of above diagram is a homotopy equivalence and we obtain a factorization of the map $\omega^\sharp \colon \Omega^2 BSU(2) \times \Omega^2 BSU(2) \to \Omega^2 \mathcal{F}$ through $\Omega^3 S^7$.

This situation is generalized by the following theorem. Let $E \colon S^6 \to \Omega S^7$ be the natural inclusion, $\beta \in \pi_7(BSU(2)) \cong \pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$ the standard generator and $r \colon \Omega S^7 \to S^3$ the composition of $\Omega\beta$ with a homotopy inverse of the standard equivalence $S^3 \simeq \Omega BSU(2)$.

Theorem 4.15 *There are natural homomorphisms γ, $\bar{\gamma}$ such that the following diagram is commutative*

$$\begin{tikzcd}
\pi_2(X) \otimes \pi_2(X) \arrow[r, "\gamma"] \arrow[d, "p"] & {[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^6]} \arrow[d, "(\Omega^2 E)\circ"] \\
\mathcal{S}^2(\pi_2(X)) \arrow[r, "\bar{\gamma}"] \arrow[d, "{[\cdot,\cdot]}"] & {[\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7]} \arrow[d, "(\Omega^2 r)\circ"] \\
\pi_3(X) \arrow[r, "\sharp"] & {[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3]}
\end{tikzcd}$$

where p is the projection and $[\cdot,\cdot]$ the Whitehead product.

Proof: Consider the map $\wedge\colon \Omega S^3 \times \Omega S^3 \to \Omega^2 S^6$, given by the formula $(f,g) \mapsto f \wedge g$. For any space T, composition on the right with $\wedge$ induces a bilinear map

$$\wedge^\sharp\colon [T,\Omega S^3] \times [T,\Omega S^3] \longrightarrow [T,\Omega^2 S^6]$$

and the composition

$$(E \circ \wedge)^\sharp\colon [T,\Omega S^3] \times [T,\Omega S^3] \longrightarrow [T,\Omega^2 S^6] \longrightarrow [T,\Omega^3 S^7]$$

is bilinear and symmetric. The homotopy equivalence $\Omega^2 BSU(2) \cong \Omega S^3$ is an H-map, so it induces a group isomorphism

$$i\colon [T,\Omega^2 BSU(2)] \cong [T,\Omega S^3].$$

Next observe that the map $\pi_2(X) \times \mathrm{Maps}(X, BSU(2)) \to \Omega^2 BSU(2)$ given by composition induces a group homomorphism

$$g\colon \pi_2(X) \longrightarrow [\mathrm{Maps}(X, BSU(2)), \Omega^2 BSU(2)].$$

This shows that, for $T = \mathrm{Maps}(X, BSU(2))$, the composition

$$\wedge^\sharp \circ \big((i \circ g) \times (i \circ g)\big)\colon \pi_2(X) \times \pi_2(X) \longrightarrow [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^6]$$

is bilinear, so it gives rise to a map

$$\gamma\colon \pi_2(X) \otimes \pi_2(X) \longrightarrow [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^6].$$

Composing γ with $\Omega^2 E$ gives a symmetric map

$$\Omega^2 E \circ \gamma\colon \pi_2(X) \otimes \pi_2(X) \longrightarrow [\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7],$$

so this map factors in a unique way through $\mathcal{S}^2(\pi_2(X))$, and this factorization defines the map $\bar\gamma$ and makes the top square of the diagram commutative.

In order to see the commutativity of the bottom square, observe that it is enough to show commutativity on $\pi_2(X) \times \pi_2(X)$. By naturality of the constructions it is enough to consider the universal example $X = S^2 \vee S^2$ and $\iota_1 \times \iota_2 \in \pi_2(X) \times \pi_2(X)$. For this case, the result reduces to the commutativity of the diagram of lemma 4.14, if one recalls that the composition

$$S^7 \xrightarrow{[\iota,\iota]} S^4 \xrightarrow{\iota} BSU(2)$$

is the generator $\beta \in \pi_7(BSU(2))$ (see [30]), so the composition of the right hand vertical arrows in this diagram gives precisely the map $(\Omega^2 r)\circ$. □

Theorem 4.16 $12\pi_3(X) \subseteq ker\, \natural$

Proof: Let $f\colon S^6 \to S^3$ be any map and let T be any CW complex. Consider the induced homomorphism of abelian groups

$$\bar{f}\colon [\Sigma^2 T, S^6] \longrightarrow [\Sigma^2 T, S^3].$$

Since S^3 is an H-space, the degree-k map on S^3 induces multiplication by k on $[\Sigma^2 T, S^3]$. Also since S^3 is an H-space, the composition $S^6 \xrightarrow{f} S^3 \xrightarrow{\deg k} S^3$ represents the element $k \cdot [f] \in \pi_6(S^3) \cong \mathbf{Z}/12\mathbf{Z}$. Taking $k = 12$ it follows that 12 annihilates the image of $\bar{f}$ in $[\Sigma^2 T, S^3]$. Now take T to be $\mathrm{Maps}(X, BSU(2))$ and take f to be $r \circ E$ in the diagram of theorem 4.15. Since p is surjective, it follows that $12\mathcal{S}^2(\pi_2(X)) \subseteq ker\, \natural$. All that is left to be shown is that, for $\sigma \in \pi_2(X)$, the element $[\sigma \otimes \sigma] \in \mathcal{S}^2(\pi_3(X))$, which we know to be annihilated by 12, is in fact already annihilated by 6. As universal example consider the case $X = S^2$, σ the class of the identity. It is well known that $[\sigma, \sigma] = 2\eta \in \pi_3(S^2)$. It follows from James' theorem that there is a homotopy equivalence $\Sigma\Omega S^3 \simeq \bigvee_{k=1}^{\infty} S^{2k+1}$ (see [31]). Hence we get

$$[\Omega S^3, \Omega^2 S^3] \cong [\Sigma\Omega S^3, \Omega S^3] \cong \prod_{k=2}^{\infty} \pi_{2k+1}(S^3)$$

We already know that 8 annihilates $3\eta^\sharp \in [\Omega S^3, \Omega^2 S^3]$. But, as I. James has shown, 4 annihilates the 2-primary components of $\pi_*(S^3)$ (see [6]), so 4 annihilates $3\eta^\sharp$, which means that 6 annihilates $2\eta^\sharp$. □

Proof of theorem 4.11: Let P and Q be two symmetric integer matrices and suppose they are equivalent over $\mathbf{Z}/12\mathbf{Z}$, i. e. there is a matrix $\bar{T}$ with entries in $\mathbf{Z}/12$ such that $\bar{T}^T\bar{P}\bar{T} = \bar{Q}$ where $\bar{P}$ and $\bar{Q}$ denote the mod 12 reductions of P and Q. Let U and V be the mod 3 and mod 4 reductions of $\bar{T}$. Then $\det V = \pm 1 \in \mathbf{Z}/4\mathbf{Z}$. Now it follows from corollary 3.8 that we can choose a matrix U' such that

$$\det U' = \begin{cases} 1 \in \mathbf{Z}/3\mathbf{Z} & \text{if } \det V = 1 \in \mathbf{Z}/4\mathbf{Z} \\ -1 \in \mathbf{Z}/3\mathbf{Z} & \text{if } \det V = -1 \in \mathbf{Z}/4\mathbf{Z} \end{cases}$$

and $U'^T\tilde{P}U' = \tilde{Q}$ where $\tilde{P}$ and $\tilde{Q}$ are the mod3 reductions of P and Q. Now let T' be chosen with entries in $\mathbf{Z}/12\mathbf{Z}$ such that $T' \equiv U'$ mod 3 and $T' \equiv V$ mod 4. According to the Chinese remainder theorem such a T' exists and is unique. Observe that $\det T' = \pm 1 \in \mathbf{Z}/12\mathbf{Z}$ and $T'^T\bar{P}T' = \bar{Q}$. Now, using lemma 3.10, we can choose an integer lift T of T' such that $\det T = \pm 1 \in \mathbf{Z}$. Then clearly $T^TPT \equiv Q$ mod 12. Now choose a homotopy equivalence $\tau\colon S^2 \vee \ldots \vee S^2 \to S^2 \vee \ldots \vee S^2$ inducing the map given by the matrix T in cohomology. Then, on $\pi_3(S^2 \vee \ldots \vee S^2)$, $\tau_*(f) \equiv g$ (mod 12). From theorem 4.16 it follows that $(\tau_*(f))^\sharp$ is homotopic to $g^\sharp$ and the two homotopy fibres $\mathrm{Maps}(Y, BSU(2))$ and $\mathrm{Maps}(Z, BSU(2))$ are homotopy equivalent. □

In the following, let $S^3_{<n>}$ denote the n-connected cover of S^3, let $\xi\colon S^3_{<5>} \to S^3$ be the canonical map and let

$$\xi\colon [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3_{<5>}] \longrightarrow [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3]$$

be the map given by composition on the left with $\Omega^2\xi$. Notice that, since ΩS^7 is 5-connected, we can choose a map $q\colon \Omega S^7 \to S^3_{<5>}$ such that $r \simeq \xi \circ q$. For the following, identify $\pi_3(X)$ with $\Gamma_2(H_2(X))$ and $\pi_4(\Sigma X)$ with $H_2(X) \otimes \mathbf{Z}/2\mathbf{Z}$ via the natural isomorphisms.

Theorem 4.17 *There are natural maps*

$$\theta_1 : [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3] \longrightarrow H_2(X) \otimes \mathbf{Z}/2\mathbf{Z},$$

$$\theta_2 : [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3_{<5>}] \longrightarrow \Gamma_2(H_2(X)) \otimes \mathbf{Z}/12\mathbf{Z}$$

and

$$\theta_3 : [\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7] \longrightarrow \Gamma_2(H_2(X))$$

such that

i) *The following diagram is commutative, where the right hand vertical arrow is the natural projection.*

$$\begin{tikzcd}
{[\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7]} \arrow[r, "\theta_3"] \arrow[d, "(\Omega^2 q)\circ"] & \Gamma_2(H_2(X)) \arrow[d, "p"] \\
{[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3_{<5>}]} \arrow[r, "\theta_2"] & \Gamma_2(H_2(X)) \otimes \mathbf{Z}/12\mathbf{Z}
\end{tikzcd}$$

ii) *The following diagram is commutative, where the vertical arrows describe the short exact sequence given by the Whitehead product and the suspension map.*

$$\begin{tikzcd}
\mathcal{S}^2(\pi_2(X)) \arrow[dr, "\tilde{\gamma}"] \arrow[dd, "{[\cdot,\cdot]}"] & \\
& {[\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7]} \arrow[dl, "\theta_3"] \\
\pi_3(X) \arrow[dr, "\sharp"] \arrow[dd, "E"] & \\
& {[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3]} \arrow[dl, "\theta_1"] \\
\pi_4(\Sigma X) &
\end{tikzcd}$$

iii) *The map θ_2 factorizes through image $\tilde{\xi} \subset [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3]$.*

Proof: The classifying map of the standard generator of $H^4(\Omega^3 S^7)$ induces a map $[\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7] \to [\mathrm{Maps}(X, BSU(2)), K(\mathbf{Z}, 4)] \cong \Gamma_2(H_2(X))$. Define θ_3 to be this composition. In the same way we obtain θ_2 induced by the map $\Omega^2 S^3_{<5>} \to K(\mathbf{Z}/12\mathbf{Z}, 4)$. For the first statement of the theorem observe that the map q induces a surjection on π_6. Hence the following diagram is commutative, where the right hand vertical arrow is the natural projection.

$$\begin{tikzcd}
\Omega^3 S^7 \arrow[r, dash] \arrow[d, "\Omega^2 q"] & K(\mathbf{Z}, 4) \arrow[d] \\
\Omega^2 S^3_{<5>} \arrow[r] & K(\mathbf{Z}/12\mathbf{Z}, 4)
\end{tikzcd}$$

This immediately implies the result. In order to define θ_1, recall from section 4.2 that

$$[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3] \cong [\mathrm{Maps}(X, BSU(2)), \Omega^2 \mathcal{F}].$$

Now define θ_1 using the map $\Omega^2 \mathcal{F} \to K(\mathbf{Z}/2\mathbf{Z}, 2)$.

For the second statement, in order to prove the commutativity of the top triangle, notice that because of the naturality of the constructions it is sufficient to prove the statement for the universal example $X = S^2 \vee S^2$ and the element $[\iota_1 \otimes \iota_2] \in \mathcal{S}^2(\pi_2(X))$. Its image $[\iota_1, \iota_2] \in \pi_3(S^2 \vee S^2)$ corresponds to the element $(1, 0) \cdot (0, 1) \in \Gamma_2(\mathbf{Z} \oplus \mathbf{Z})$. Now recall from the construction of the map $\bar{\gamma}$ (theorem 4.15) that

$$\bar{\gamma}([\iota_1 \otimes \iota_2]) \colon \Omega S^3 \times \Omega S^3 \longrightarrow \Omega^3 S^7$$

is the map that sends (f, g) to $\Sigma f \wedge g$. Now notice that if one suspends the map

$$(\Sigma^2 g) \circ f \colon S^1 \longrightarrow S^5$$

twice, it becomes canonically homotopic to $\Sigma f \wedge g$. It follows that the map $\bar{\gamma}([\iota_1 \otimes \iota_2])$ is homotopic to the composition

$$\Omega S^3 \times \Omega S^3 \xrightarrow{\rho} \Omega S^5 \xrightarrow{E^2} \Omega^3 S^7$$

where ρ maps (f,g) to $(\Sigma^2 g)\circ f$ and E^2 is the double suspension map. Since the map E^2 induces an isomorphism on fourth cohomology, $\theta_3(\bar{\gamma}([\iota_1 \otimes \iota_2]))$ is equal to the pull-back of the standard generator $\sigma \in H^4(\Omega S^5)$ via the map ρ. Since the restriction of ρ to either of the two factors is contractible, it follows that $\rho^*(\sigma)$ is a multiple of $\beta_1\beta_2 \in H^4(\Omega S^3 \times \Omega S^3) \cong \Gamma_2(\beta_1, \beta_2)$. All we have to show is that $\rho^*(\sigma) = \beta_1\beta_2$. Now consider the following diagram, where the horizontal arrows come from the standard inclusion $\iota\colon S^k \to \Omega\Sigma S^k$ ($k \in \{2,4\}$) and the left vertical arrow is the standard projection $S^2 \times S^2 \to S^2 \wedge S^2$.

$$\begin{tikzcd} S^2 \times S^2 \arrow[r, "\iota\times\iota"] \arrow[d, dash] & \Omega S^3 \times \Omega S^3 \arrow[d, "\rho"] \\ S^4 \arrow[r, dash, "\iota"] & \Omega S^5 \end{tikzcd}$$

One readily checks that this diagram commutes. In fact, both compositions map the pair $(x,y) \in S^2 \times S^2$ to the map $t \mapsto t \wedge x \wedge y$. The pull-back of σ via the bottom and left arrows is the standard generator of $H^4(S^2 \times S^2)$. But the top arrow induces the standard isomorphism $H^2(\Omega S^3) \otimes H^2(\Omega S^3) \to H^4(S^2 \times S^2)$, hence the result.

In order to prove the commutativity of the bottom triangle of the second diagram, let $f \in \pi_3(X)$. Observe that $\theta_1(f^\sharp)$ can be described as the pull-back of the non-trivial class $u \in H^2(\Omega^2\mathcal{F}; \mathbf{Z}/2\mathbf{Z})$ via the map $f^\sharp$. Let $c \in H_2(\mathrm{Maps}(X, BSU(2)); \mathbf{Z}/2\mathbf{Z})$. Recall that $H_2(\mathrm{Maps}(X, BSU(2)))$ is naturally isomorphic to $H^2(X)$ and let $\gamma \in H^2(X; \mathbf{Z}/2\mathbf{Z})$ be the class corresponding to c. Then theorem 4.5 states that

$$\left\langle (f^\sharp)^*(u), c \right\rangle = Q_f(\gamma, \gamma)$$

where Q_f denotes the bilinear form on $H^2(X; \mathbf{Z}/2\mathbf{Z})$ corresponding to f. Let $h \in H_2(X; \mathbf{Z}/2\mathbf{Z})$ be the class corresponding to $(f^\sharp)^*(u)$. Then $\langle \gamma, h \rangle = Q_f(\gamma, \gamma)$ for all $\gamma \in H^2(X; \mathbf{Z}/2\mathbf{Z})$. This property characterizes h uniquely and according to lemma 2.5 u is equal to the image of $\Sigma f \in \pi_4(\Sigma X)$ under the natural identification $\pi_4(\Sigma X) \cong H_2(X;\mathbf{Z}/2\mathbf{Z})$, hence the diagram commutes.

In order to prove the third statement it is sufficient to show that the restriction of θ_2 to the kernel of $\bar{\xi}$ is zero. Abbreviate $\mathrm{Maps}(X, BSU(2))$ by $\boldsymbol{S}$. The map $\bar{\xi}$ can be written as the composition

$$[\boldsymbol{S}, \Omega^2 S^3_{<5>}] \xrightarrow{u} [\boldsymbol{S}, \Omega^2 S^3_{<4>}] \xrightarrow{v} [\boldsymbol{S}, \Omega^2 S^3_{<3>}] \xrightarrow{w} [\boldsymbol{S}, \Omega^2 S^3]$$

and we already know that the map w is an isomorphism. From the fibration

$$K(\mathbf{Z}/2\mathbf{Z}, 1) \longrightarrow \Omega^2 S^3_{<4>} \longrightarrow \Omega^2 S^3_{<3>}$$

we obtain the exact sequence

$$[\boldsymbol{S}, K(\mathbf{Z}/2\mathbf{Z}, 1)] \longrightarrow [\boldsymbol{S}, \Omega^2 S^3_{<4>}] \xrightarrow{v} [\boldsymbol{S}, \Omega^2 S^3_{<3>}]$$

and since $[\boldsymbol{S}, K(\mathbf{Z}/2\mathbf{Z}, 1)] \cong H^1(\boldsymbol{S}; \mathbf{Z}/2\mathbf{Z}) = 0$, we see that the map v is injective, so the kernel of $\bar{\xi}$ is equal to the kernel of u. Similarly, we conclude from the fibration

$$K(\mathbf{Z}/2\mathbf{Z}, 2) \xrightarrow{\delta} \Omega^2 S^3_{<5>} \longrightarrow \Omega^2 S^3_{<4>}$$

that the kernel of u is equal to the image of the map

$$\bar{\delta}\colon [\boldsymbol{S}, K(\mathbf{Z}/2\mathbf{Z}, 2)] \longrightarrow [\boldsymbol{S}, \Omega^2 S^3_{<5>}]$$

induced by composition with δ. Therefore it is enough to show that $\theta_2 \circ \bar{\delta} = 0$. Now $[\boldsymbol{S}, K(\mathbf{Z}/2\mathbf{Z}, 2)] \cong H^2(\boldsymbol{S}; \mathbf{Z}/2\mathbf{Z}) \cong H_2(X) \otimes \mathbf{Z}/2\mathbf{Z}$, which means that $\theta_2 \circ \bar{\delta}$ is a natural homomorphism from $H_2(X) \otimes \mathbf{Z}/2\mathbf{Z}$ to $\Gamma_2(H_2(X)) \otimes \mathbf{Z}/12\mathbf{Z}$. One checks that we obtain a natural transformation T from the functor $H_2(\,\cdot\,) \otimes \mathbf{Z}/2\mathbf{Z}$ to the functor $\Gamma_2(H_2(\,\cdot\,)) \otimes \mathbf{Z}/12\mathbf{Z}$. Take $X = S^2$ and suppose that $\mathsf{T}(X)$ is non-zero. Let $\beta \in H_2(S^2)$ be the standard generator. Then $\Gamma_2(H_2(S^2)) \otimes \mathbf{Z}/12\mathbf{Z}$ is generated by $\frac{\beta^2}{2}$ and the only possible non-trivial map sends $\beta \in H_2(S^2; \mathbb{Z}/2\mathbb{Z})$ to $6\frac{\beta^2}{2} = 3\beta^2 \in \Gamma_2(H_2(S^2)) \otimes \mathbb{Z}/12\mathbb{Z}$. Now take $X' = S^2 \vee S^2$ and let β_1, β_2 be the standard generators of $H_2(X')$. By naturality, considering the two standard inclusions $S^2 \to S^2 \vee S^2$, we see that $\mathsf{T}(X')(\beta_i) = 3\beta_i^2$. Considering the pinching map $S^2 \to S^2 \vee S^2$ we see that now β is mapped to $\beta_1 + \beta_2$, so $3\beta^2$ maps to $3(\beta_1 + \beta_2)^2 = 3\beta_1^2 + 3\beta_2^2 + 6\beta_1\beta_2$. Since $0 \neq 6\beta_1\beta_2 \in \Gamma_2(H_2(S^2 \vee S^2)) \otimes \mathbb{Z}/12\mathbb{Z}$, this is a contradiction to the linearity of the map $\mathsf{T}(X')$. This shows that $\mathsf{T}(S^2) = 0$. Hence, by universal example, $\mathsf{T} = 0$. This is all we had to show. □

We are now in the position to determine the kernel of the map

$$\sharp : \pi_3(X) \to [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3].$$

Theorem 4.18 $ker\, \sharp = 12\pi_3(X)$

Proof: According to theorem 4.16, $ker\, \sharp \supseteq 12\pi_3(X)$. Now we show that $ker\, \sharp \subseteq 12\pi_3(X)$. First it follows from theorem 4.17*ii* that $ker\, \sharp \subseteq ker\, E = \mathcal{S}^2(\pi_2(X))$. Now consider the following commutative diagram, obtained from theorems 4.15 and 4.17*i*.

$$\begin{tikzcd}
\mathcal{S}^2(\pi_2(X)) \arrow[r, "\tilde{\gamma}"] \arrow[dd, hook, "{[\cdot,\cdot]}"] & {[\mathrm{Maps}(X, BSU(2)), \Omega^3 S^7]} \arrow[r, "\theta_3"] \arrow[d, "\circ\Omega^2 q"] & \Gamma_2(H_2(X)) \arrow[d, "p"] \\
 & {[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3_{<5>}]} \arrow[r, "\theta_2"] \arrow[d, "\tilde{\xi}"] & \Gamma_2(H_2(X)) \otimes \mathbb{Z}/12\mathbb{Z} \\
\pi_3(X) \arrow[r, "\sharp"] & {[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^3]} &
\end{tikzcd}$$

Together with theorem 4.17*iii* it follows that $ker\, \sharp \subseteq ker(p \circ \theta_3 \circ \tilde{\gamma})$. Recall from theorem 4.17*ii* that the map $\theta_3 \circ \tilde{\gamma}$ is equal to $[\cdot, \cdot]$ under the canonical identification of $\Gamma_2(H_2(X))$ with $\pi_3(X)$. Hence $ker\, \sharp \subseteq ker\, p \cong 12\pi_3(X)$. □

Chapter 5

The Homology of the Mapping Space

5.1 Cotor and the Eilenberg-Moore Spectral Sequence

Let $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$. In the present chapter we compute the homology of $\mathrm{Maps}(Y, BSU(2))$. We use the *Eilenberg-Moore spectral sequence* (see [21]) of the fibration

$$\mathrm{Maps}_0(Y, BSU(2)) \longrightarrow \mathrm{Maps}(X, BSU(2)) \xrightarrow{f^!} \Omega^2\mathcal{F}.$$

All homology is to be taken with coefficients in a field k. Suppose $p\colon E \to B$ is a fibration and B is simply-connected. Suppose further that $f\colon X \to B$ is a continuous map and $p_f\colon E_f \to X$ is the pullback fibration of p via f. S. Eilenberg and J. C. Moore ([13]) identified the homology of E_f in terms of the chains on B, X, E and the maps p_* and f_*.

Theorem 5.1

$$H_*(E_f, k) \cong \mathrm{Cotor}^{C_*(B;k)}(C_*(X;k), C_*(E;k))$$

The functor Cotor is the derived functor of the cotensor product. Details are given below. There is the following purely algebraic theorem.

Theorem 5.2 *Let Γ be a differential graded k-coalgebra and M, N differential graded comodules over Γ. There is a spectral sequence with*

$$E^2 \cong \mathrm{Cotor}^{H_*(\Gamma)}(H_*(M), H_*(N))$$

which, if it is convergent, converges to $\mathrm{Cotor}^{\Gamma}(M, N)$.

If B is connected and simply-connected, this spectral sequence for $\Gamma = C_*(B)$, $M = C_*(X)$ and $N = C_*(E)$ converges, where C_* denotes singular chains with k-coefficients. Combining the two theorems one obtains the following.

Theorem 5.3 ('Eilenberg-Moore Spectral Sequence') *Let $p\colon E \to B$ be a fibration with simply-connected base, $f\colon X \to B$ a continuous map and E_f the total space of the pull-back fibration of p via f. Let k be a field. Then there is a spectral sequence with*

$$E^2 \cong \mathrm{Cotor}^{H_*(B;k)}(H_*(X;k), H_*(E;k))$$

converging to $H_*(E_f;k)$.

For X a point, E_f is just the fibre of p. Hence we can compute the homology of the fibre of a fibration via the Eilenberg-Moore spectral sequence.

Corollary 5.4 *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration with simply-connected base and k a field. Then there is a spectral sequence with*

$$E^2 \cong \mathrm{Cotor}^{H_*(B;k)}(k, H_*(E;k))$$

converging to $H_*(F)$.

Now we briefly summarize the background, details can be found in [21] or [13]. Let V be a topological space. Recall that the diagonal map composed with the Alexander-Whitney map induces a coproduct

$$\Delta\colon C_*(V) \longrightarrow C_*(V \times V) \longrightarrow C_*(V) \otimes C_*(V)$$

which makes $C_*(V)$ a differential graded coalgebra. Also, a continuous map $f: U \to V$ of topological spaces gives $C_*(U)$ the structure of a comodule over $C_*(V)$ via the composition

$$\lambda_f: C_*(U) \xrightarrow{\Delta} C_*(U) \otimes C_*(U) \xrightarrow{f_*\otimes \mathrm{id}} C_*(V) \otimes C_*(U).$$

The same statements hold, if one replaces (C_*, d) by $(H_*, 0)$.

Now let C be a differential graded k-coalgebra with counit $\varepsilon: C \to k$. Let A be a right C-comodule and B a left C-comodule and let $\lambda_A: A \to A \otimes_k C$ and $\lambda_B: B \to C \otimes_k B$ denote the C-coactions on A and B. We can define the *cotensor product* $A \,\square_C\, B$ as the kernel of the map

$$(\lambda_A \otimes \mathrm{id} - \mathrm{id} \otimes \lambda_B): A \otimes_k B \longrightarrow A \otimes_k C \otimes_k B.$$

Notice that, if C is co-commutative, $A \,\square_C\, B$ is a C-bicomodule, its structure as a right comodule being induced by λ_A and as a left comodule by λ_B. A map $g: A \to A'$ of graded comodules is called a *proper monomorphism*, if the following conditions are satisfied:

i) As a map of graded vector spaces, g is a monomorphism;

ii) the induced map on cycles $Z(g): Z(A) \to Z(A')$ is a monomorphism;

iii) the induced map on homology $H(g): H(A) \to H(A')$ is a monomorphism.

In the same way define *proper epimorphisms*, *proper exact sequences* etc., by imposing the three conditions analogous to those above. A differential graded comodule I is called *proper injective*, if for any morphism $\sigma: B \to I$ and proper monomorphism $i: B \to A$ the map σ factors through A, i. e. there is a map $\tilde{\sigma}: A \to I$ such that $\tilde{\sigma} \circ i = \sigma$. Define a *proper injective resolution* of the left C-comodule N to be a proper exact sequence

$$N \xrightarrow{i} X_0 \xrightarrow{\delta} X_1 \xrightarrow{\delta} X_2 \xrightarrow{\delta} X_3 \xrightarrow{\delta} \cdots$$

where X_i is a proper injective differential graded comodule for all $i \geq 0$. Let d_i denote the internal differential on X_i. The graded comodules X_i ($i \geq 0$) form a double complex by setting $X_{i,j} = (X_i)_j$ with internal differential $d_i \colon X_{i,j} \to X_{i,j-1}$ and external differential $\delta \colon X_{i,j} \to X_{i+1,j}$. Form the associated total complex $(\mathrm{Tot}(X_\bullet), D)$ by setting

$$(\mathrm{Tot}(X_\bullet))_n = \bigoplus_{j-i=n} X_{i,j}, \qquad D = \bigoplus_i (\delta + (-1)^i d_i).$$

In the obvious way, $(\mathrm{Tot}(X_\bullet), D)$ is a differential $\mathbb{Z}$-graded left C-comodule. For a right C-comodule M, form the $\mathbb{Z}$-graded differential C-bicomodule

$$(M \,\square_C\, \mathrm{Tot}(X_\bullet), d_M \,\square\, \mathrm{id} \pm \mathrm{id} \,\square\, D)$$

where the sign is $(-1)^m$ on elements in $M_m \,\square\, \mathrm{Tot}(X_\bullet)$. Notice that the cotensor product $M \,\square_C\, \_\_$ preserves the external degree of elements in $\mathrm{Tot}(X_\bullet)$, so we get a decomposition

$$\left(M \,\square_C\, \mathrm{Tot}(X_\bullet)\right)_n = \bigoplus_{i \geq -n} \left(M \,\square_C (\mathrm{Tot}(X_\bullet))_{i,*}\right)_n$$

which induces a bigrading

$$\left(M \,\square_C\, \mathrm{Tot}(X_\bullet)\right)_{i,j} = \left(M \,\square_C (\mathrm{Tot}(X_\bullet))_{i,*}\right)_{j-i}$$

Now define $\mathrm{Cotor}^C(M, N)$ as

$$H\left(M \,\square_C\, \mathrm{Tot}(X_\bullet), d_M \,\square\, \mathrm{id} \pm 1 \,\square\, D\right).$$

One can show that proper injective resolutions always exist, and that the definition of Cotor is independent of the choice of the resolution. Also, resolving M by proper injective right C-comodules and forming the cotensor product with N leads to the same result. Notice that $\mathrm{Cotor}^C(M, N)$ inherits the bigrading from $M \,\square_C\, \mathrm{Tot}(X_\bullet)$ and $\mathrm{Cotor}^C_{0,*}(M, N) = M \,\square_C\, N$.

A coalgebra C is called *connected* if the counit $\varepsilon\colon C \to k$ is an isomorphism in degree zero, and *simply connected* if it contains no elements of degree one. Recall that, if B is a connected and simply-connected space, there is a natural way of constructing a connected, simply-connected differential graded coalgebra $\bar{C}_*(B)$ and a chain equivalence $C_*(B) \to \bar{C}_*(B)$ (see [1]). In the following we will without loss of generality assume that whenever a space B is connected and simply connected the same is true for $C_*(B)$. The following is proved in [13].

Lemma 5.5 *Let C_1, C_2 be connected, simply connected differential graded coalgebras and let A_i, B_i be differential graded C_i-comodules for $i \in \{1, 2\}$. Then there is a natural isomorphism*

$$\mathrm{Cotor}^{C_1 \otimes C_2}(A_1 \otimes A_2, B_1 \otimes B_2) \xrightarrow{\cong} \mathrm{Cotor}^{C_1}(A_1, B_1) \otimes \mathrm{Cotor}^{C_2}(A_2, B_2)$$

As a consequence one obtains, under the hypotheses of theorem 5.1, a natural coproduct given by the composition

$$\begin{aligned}
&\mathrm{Cotor}^{C_*(B;k)}\big(C_*(X;k), C_*(E;k)\big) \\
&\xrightarrow{\Delta} \quad \mathrm{Cotor}^{C_*(B\times B;k)}\big(C_*(X\times X;k), C_*(E\times E;k)\big) \\
&\xrightarrow{\text{A.-W.}} \quad \mathrm{Cotor}^{C_*(B;k)\otimes C_*(B;k)}\big(C_*(X;k)\otimes C_*(X;k), C_*(E;k)\otimes C_*(E;k)\big) \\
&\xrightarrow{\cong} \quad \mathrm{Cotor}^{C_*(B;k)}\big(C_*(X;k), C_*(E;k)\big) \otimes \mathrm{Cotor}^{C_*(B;k)}\big(C_*(X;k), C_*(E;k)\big)
\end{aligned}$$

and one can show that the isomorphism of theorem 5.1 is an isomorphism of coalgebras. Furthermore, the Eilenberg-Moore spectral sequence is a spectral sequence of coalgebras, converging to its target as a coalgebra (see [21]). Also, for any connected, simply-connected k-coalgebra C, lemma 5.5 gives rise to a coproduct on $\mathrm{Cotor}^C(k,k)$ via the composition

$$\mathrm{Cotor}^C(k,k) \to \mathrm{Cotor}^{C\otimes C}(k\otimes k, k\otimes k) \to \mathrm{Cotor}^C(k,k) \otimes \mathrm{Cotor}^C(k,k).$$

One important example of a proper injective resolution is the *cobar resolution*. Suppose C is connected. Let $\overline{C} = ker\ \varepsilon = \{c \in C \mid \deg c > 0\}$ and define for $n \geq 0$

$$F_n(C, N) = C \otimes_k \underbrace{\overline{C} \otimes_k \cdots \otimes_k \overline{C}}_{n \text{ times}} \otimes_k N$$

The coproduct on C gives $F_n(C, N)$ the structure of a left C-comodule. Let d_C denote the differential on C and also its restriction to $\overline{C}$, and d_N the differential on N. Write a typical simple tensor in $F_n(C, N)$ as $c\,[\,c_1\,|\,c_2\,|\,\ldots\,|\,c_n\,]\,a$ where $c \in C$, $c_i \in \overline{C}$ and $a \in N$, and write $\overline{c}$ for $(-1)^{1+\deg c}c$. Define an internal differential $d_n\colon F_n(C, N)_j \to F_n(C, N)_{j-1}$ by the formula

$$\begin{aligned} d_n(c\,[\,c_1\,|\,c_2\,|\,\ldots\,|\,c_n\,]\,a) &= d_C(c)\,[\,c_1\,|\,c_2\,|\,\ldots\,|\,c_n\,]\,a \\ &\quad + \sum_{i=1}^{n} \overline{c}\,[\,\overline{c}_1\,|\,\ldots\,|\,\overline{c}_{i-1}\,|\,d_C(c_i)\,|\,c_{i+1}\,|\,\ldots\,|\,c_n\,]\,a \\ &\quad + \overline{c}\,[\,\overline{c}_1\,|\,\ldots\,|\,\overline{c}_n\,]\,d_N(a) \end{aligned}$$

One checks that this gives $F_n(C, N)$ the structure of a differential graded C-comodule. Let Δ denote the coproduct on C as well as its restriction to $\overline{C}$ and λ_N the C-coaction on N. For any graded module B let $T\colon B \to B$ be the map that sends b to $\overline{b}$. Let $\overline{\Delta} = (T \otimes \mathrm{id}) \circ \Delta$ and $\overline{\lambda}_N = (T \otimes \mathrm{id}) \circ \lambda_N$. Define an external differential $\delta\colon F_n(C, N) \to F_{n+1}(C, N)$ by the formula

$$\begin{aligned} \delta(c\,[\,c_1\,|\,c_2\,|\,\ldots\,|\,c_n\,]\,a) &= \overline{\Delta}(c)\,|\,c_1\,|\,c_2\,|\,\ldots\,|\,c_n\,]\,a \\ &\quad + \sum_{i=1}^{n} \overline{c}\,[\,\overline{c}_1\,|\,\ldots\,|\,\overline{c}_{i-1}\,|\,\overline{\Delta}(c_i)\,|\,c_{i+1}\,|\,\ldots\,|\,c_n\,]\,a \\ &\quad + \overline{c}\,[\,\overline{c}_1\,|\,\ldots\,|\,\overline{c}_n\,|\,\overline{\lambda}_N(a) \end{aligned}$$

One checks that $\delta \circ \delta = 0$, $\delta \circ d_n + d_{n+1} \circ \delta = 0$ and that (taking the internal differential to be $(-1)^{i+1}d_i$) the complex $(F_*(C, N), \delta)$, together with the map $\overline{\lambda}_N\colon N \to F_0(C, N)$, forms a proper injective resolution of N. Let

$$F_n(M, C, N) = M \,\square_C\, F_n(C, N) = M \otimes_k \underbrace{\overline{C} \otimes_k \cdots \otimes_k \overline{C}}_{n \text{ times}} \otimes_k N$$

with differential $D = d_M \square \mathrm{id}_F \pm \mathrm{id}_M \square (d + \delta)$. Then, according to the definitions,

$$\mathrm{Cotor}^C(M, N) = H\Big(F(M, C, N), D\Big).$$

Observe that the connectivity hypotheses about C ensure that there are no elements of negative total degree in the cobar resolution.

Now consider the calculation of $\mathrm{Cotor}^C(k, k)$. On the comodule $F_*(k, C, k)$ there is a natural *juxtaposition product*

$$m\colon F_i(k, C, k) \otimes F_j(k, C, k) \longrightarrow F_{i+j}(k, C, k)$$

given by

$$[\, c_1 \,|\, \ldots \,|\, c_i \,] \otimes [\, c_{i+1} \,|\, \ldots c_{i+j} \,] \longmapsto [\, c_1 \,|\, \ldots \,|\, c_{i+j} \,].$$

Furthermore, if C is co-commutative, then the comodule $F_*(k, C, k)$ carries a natural coalgebra structure, inducing the coproduct on $\mathrm{Cotor}^C(k, k)$. Let Σ_{p+q} denote the group of permutations of the set $\{1, \ldots, p+q\}$. A permutation $\sigma \in \Sigma_{p+q}$ is called a (p, q)-*shuffle*, if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2), \cdots < \sigma(p+q).$$

Let $\Sigma_{p,q}$ denote the set of (p, q)-shuffles. Define the *shuffle coproduct*

$$\Delta\colon F_*(k, C, k) \to F_*(k, C, k) \otimes F_*(k, C, k)$$

by the formula

$$\Delta([\, c_1 \,|\, \ldots \,|\, c_n \,]) = \sum_{\substack{p+q=n \\ \sigma \in \Sigma_{p,q}}} (-1)^{s(\sigma)} [\, c_{\sigma(1)} \,|\, \ldots \,|\, c_{\sigma(p)} \,] \otimes [\, c_{\sigma(p+1)} \,|\, \ldots \,|\, c_{\sigma(p+q)} \,].$$

Here $s(\sigma) = \sum (1 + \deg c_i)(1 + \deg c_j)$ where the sum is taken over all pairs (i, j) with $i < j$ and $\sigma(i) > \sigma(j)$. The following is proved in [21].

Theorem 5.6 *If C is a graded co-commutative differential k-coalgebra, then, with multiplication m and comultiplication Δ defined as above, $F_*(k, C, k)$ is* *a differential graded Hopf algebra and this induces a Hopf algebra structure on* $\mathrm{Cotor}^C(k,k)$. *Furthermore, the induced coproduct agrees with the one obtained from lemma 5.5.*

Example 5.7 Consider the exterior Hopf algebra $\Lambda\{x\}$ in one primitive generator x, $\deg x > 1$, and zero differential. We obtain $F_*(k, \Lambda\{x\}, k) \cong k[y]$, the polynomial algebra generated by the element $y = [x] \in F_1(k, \Lambda\{x\}, k)$ of bidegree $(\deg x, 1)$. The internal differential is zero, because it is induced by the differential on $\Lambda\{x\}$. The external differential preserves the internal degree, hence it is zero and we obtain, as Hopf algebras,

$$\mathrm{Cotor}^{\Lambda\{x\}}(k,k) \cong F_*(k, \Lambda\{x\}, k) \cong k[y]$$

where $y = [x] \in \mathrm{Cotor}^{\Lambda\{x\}}_{\deg x, 1}(k,k)$. For the coproduct of y^2 we obtain

$$\Delta y^2 = y^2 \otimes 1 + 1 \otimes y^2 + y \otimes y + (-1)^{1+\deg x} y \otimes y,$$

so if we suppose that $\mathrm{char}\, k = p$ a prime and that either $p = 2$ or $\deg x$ is odd then the primitives are precisely the elements of the form $y^{(p^n)}$ for $n \in \mathbf{N}$.

Example 5.8 Let $\mathrm{char}\, k = p$ an odd prime and let $C = k[x]/x^p$ where x is primitive and $\deg x > 0$ is even. Let $Q = C \otimes \Lambda\{u\} \otimes k[y]$ be the bigraded k-algebra with $\mathrm{bideg}\, x = (\deg x, 0)$, $\mathrm{bideg}\, u = (\deg x, 1)$ and $\mathrm{bideg}\, y = (p \cdot \deg x, 2)$. Q becomes a Hopf algebra by declaring x, u and y to be primitives. Q gets the structure of a C-comodule via $\Delta_C \otimes \mathrm{id}_{\Lambda\{u\}} \otimes \mathrm{id}_{k[y]}$ and one checks that $Q_{*,j}$ is an injective C-comodule. Introduce a differential $\delta \colon Q_{*,j} \to Q_{*,j+1}$ by

$$\begin{aligned} \delta(x^n \otimes 1 \otimes y^m) &= n \cdot x^{n-1} \otimes u \otimes y^m \\ \delta(x^n \otimes u \otimes y^m) &= \begin{cases} 1 \otimes 1 \otimes y^{m+1} & \text{if } n = p-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

One checks that δ is a map of comodules and that the sequence

$$0 \longrightarrow k \longrightarrow Q_{*,0} \longrightarrow Q_{*,1} \longrightarrow Q_{*,2} \longrightarrow \cdots$$

is exact, so Q is an injective resolution of k. We obtain $k \,\square_C\, Q \cong \Lambda\{u\} \otimes k[\,y\,]$ and the differential vanishes for dimension reasons. Therefore we obtain

$$\mathrm{Cotor}^C(k, k) \cong \Lambda\{u\} \otimes k[\,y\,]$$

where $u \in \mathrm{Cotor}^C_{\deg x,1}(k, k)$ and $y \in \mathrm{Cotor}^C_{p\cdot\deg x,2}(k, k)$. So far this is only an isomorphism of vector spaces. However, detecting the elements u and y in the cobar resolution, one obtains that u is represented by $[\,x\,] \in F_1(k, C, k)$ and y by $\sum_{n=1}^{p-1} \frac{1}{p}\binom{p}{n}\,[\,x^n \,|\, x^{p-n}\,]$ where, for $0 < n < p$, the expression $\frac{1}{p}\binom{p}{n}$ stands for the product of $\frac{1}{p}\binom{p}{n} \in \mathbf{Z}$ and $1 \in k$. Using the cobar construction one then checks that the above isomorphism is an isomorphism of Hopf algebras.

5.2 The Homology of Loop Spaces

We record some well-known facts about the homology of loop spaces. They can be found in [7], see also [6] and [19]. For any space T there are operations

$$Q_i \colon H_q(\Omega^n T; \mathbf{Z}/2\mathbf{Z}) \longrightarrow H_{2q+i}(\Omega^n T; \mathbf{Z}/2\mathbf{Z}) \qquad (0 \leq i \leq n-1)$$

which are natural for n-fold loop maps and linear for $i < n-1$. For $I = (i_1, \ldots, i_j)$, $j \geq 0$, let $Q_I = Q_{i_1} Q_{i_2} \cdots Q_{i_j}$, $\lambda(I) = i_j$ and $\ell(I) = j$. In the case $j = 0$, Q_I is understood to be the identity map. I is called *admissible*, if $0 < i_1 \leq i_2 \leq \cdots \leq i_j$. For $s \geq 1$, $n \geq 1$ let $x_n \in H_n(\Omega^s S^{s+n})$ be the image of the fundamental class of S^n under the suspension map E^s.

Theorem 5.9

i) *For $s, n \geq 1$ there is an isomorphism of Hopf algebras*

$$\bigotimes_{\substack{I \text{ admissible} \\ \lambda(I) \leq s-1}} \mathbf{Z}/2\mathbf{Z}[\,Q_I x_n\,] \xrightarrow{\cong} H_*(\Omega^s S^{n+s}; \mathbf{Z}/2\mathbf{Z})$$

where the $Q_I x_n$ are primitive.

ii) *For $n \geq 1$ there is an isomorphism of algebras*

$$\bigotimes_{\substack{I \text{ admissible} \\ \lambda(I)\leq n-1,\, \ell(I)\geq 1}} \mathbb{Z}/2\mathbb{Z}[Q_I[1]] \xrightarrow{\cong} H_*(\Omega^n_0 S^n; \mathbb{Z}/2\mathbb{Z})$$

where $[1] \in H_0(\Omega^n S^n)$ is the class of the non-basepoint and $Q_I[1]$ stands for $Q_I[1] \cdot [-2^{\ell(I)}]$.

Theorem 5.10 *Let p be an odd prime and $k = \mathbb{Z}/p\mathbb{Z}$.*

i) *As a Hopf algebra, $H_*(\Omega^2 S^3; k)$ is isomorphic to*

$$\bigotimes_{i\geq 0} \Lambda\{u_i\} \otimes \bigotimes_{i \geq 0} k[y_i]$$

where u_i is primitive of degree $2p^i - 1$ and y_i primitive of degree $2p^{i+1} - 2$.

ii) *As an algebra, $H_*(\Omega^3_0 S^3; k)$ is isomorphic to*

$$\bigotimes_{i>0} k[z_i] \otimes \bigotimes_{i,j\geq 0} \Lambda\{u_{i,j}\} \otimes \bigotimes_{i,j\geq 0} k[y_{i,j}]$$

where $\deg z_i = 2p^i - 2$, $\deg u_{i,j} = p^j(2p^{i+1} - 2) - 1$ and $\deg y_{i,j} = p^{j+1}(2p^{i+1} - 2) - 2$.

The following two corollaries are well known.

Corollary 5.11

i) *For $n, s \geq 1$ and taking $\mathbb{Z}/2\mathbb{Z}$-coefficients, the Eilenberg-Moore spectral sequences of the path fibrations $\Omega^s S^{n+s} \longrightarrow * \longrightarrow \Omega^{s-1} S^{n+s}$ and $\Omega^n_0 S^n \longrightarrow * \longrightarrow \Omega^{n-1} S^n_{<n>}$ collapse at E^2.*

ii) *Taking $\mathbb{Z}/p\mathbb{Z}$-coefficients for an odd prime p, the Eilenberg-Moore spectral sequence of the path fibration $\Omega^3_0 S^3 \longrightarrow * \longrightarrow \Omega^2 S^3_{<3>}$ collapses at E^2.*

Proof: If B is the base space of the respective fibration, the E^2 term of the Eilenberg-Moore spectral sequence is $\mathrm{Cotor}^{H_*(B)}(k,k)$. Notice that, using the examples 5.7 and 5.8, we can compute this E^2 term and a count of dimensions gives the result. We perform the calculation explicitly for the fibration $\Omega_0^3 S^3 \longrightarrow * \longrightarrow \Omega^2 S^3_{<3>}$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$ for p an odd prime. Notice that, as coalgebras,

$$k[\,y_i\,] \cong \bigotimes_{j=0}^{\infty} k[\,y_i^{(p^j)}\,]\Big/ y_i^{(p^{j+1})}$$

so, using example 5.8 and theorem 5.5, we get

$$\mathrm{Cotor}^{k[\,y_i\,]}(k,k) \cong \bigotimes_{j\geq 0} \Lambda\{u_{i,j}\} \otimes k[\,y_{i,j}\,]$$

where $\deg u_{i,j} = \deg y_i^{(p^j)} - 1$ and $\deg y_{i,j} = p \cdot \deg y_i^{(p^j)} - 2$. According to example 5.7, $\mathrm{Cotor}^{\Lambda\{u_i\}}(k,k) \cong k[\,z_i\,]$ where $\deg z_i = \deg u_i - 1$. From theorem 5.10 and lemma 4.2 we obtain that, as Hopf algebras,

$$H_*(\Omega^2\mathcal{F};k) \cong \bigotimes_{i\geq 1} \Lambda\{u_i\} \otimes \bigotimes_{i\geq 0} k[\,y_i\,]$$

so

$$\mathrm{Cotor}^{H_*(\Omega^2\mathcal{F};k)}(k,k) \cong \bigotimes_{i\geq 1} k[\,z_i\,] \otimes \bigotimes_{i\geq 0}\bigotimes_{j\geq 0}\Big(\Lambda\{u_{i,j}\} \otimes k[\,y_{i,j}\,]\Big)$$

where

$$\deg z_i = \deg u_i - 1 = 2p^i - 2,$$

$$\deg u_{i,j} = \deg y_i^{(p^j)} - 1 = p^j(2p^{i+1} - 2) - 1,$$

$$\deg y_{i,j} = p \cdot \deg y_i^{(p^j)} - 2 = p^{j+1}(2p^{i+1} - 2) - 2.$$

Comparing this E^2 term with the target of the spectral sequence, we see that they are isomorphic, so the spectral sequence collapses. The remaining cases can be done similarly. $\square$

Corollary 5.12 *For $s, n \geq 1$, the suspension maps $\Omega^s S^{n+s} \to \Omega^{s+1} S^{n+s+1}$ and $\Omega_0^s S^s \to \Omega_0^{n+s} S^{n+s}$ induce monomorphisms in homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients.*

Proof: In homology $x_n \in H_n(\Omega^s S^{n+s})$ is mapped to $x_n \in H_n(\Omega^{s+1} S^{n+s+1})$ and $[1] \in H_0(\Omega_0^n S^n)$ to $[1] \in H_0(\Omega_0^{n+1} S^{n+1})$. The maps in homology are induced by s-fold loop maps, so they commute with the operations Q_i for $i \leq s-1$, and the result follows. □

5.3 The Z/2Z-Homology

Let $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$. We calculate $H_*(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/2\mathbb{Z})$ in two steps. First we compute the E^2-term of the Eilenberg-Moore spectral sequence and then we show its collapsing at E^2. We need the structure of $H_*(\mathrm{Maps}(X, BSU(2)); \mathbb{Z}/2\mathbb{Z})$ as a comodule over $H_*(\Omega^2\mathcal{F}; \mathbb{Z}/2\mathbb{Z})$, so we determine the induced map $f_*^\sharp$ in $\mathbb{Z}/2\mathbb{Z}$-homology. It is clear that the map $f_*^\sharp$ in homology preserves the coproduct. However, since $f^\sharp$ is in general not a map of H-spaces (not even localized at the prime 2), one should not, in general, expect $f_*^\sharp$ to be compatible with the Pontryagin products. Nevertheless, in $\mathbb{Z}/2\mathbb{Z}$ homology this is the case.

Theorem 5.13 *The map $f_*^\sharp$ in $\mathbb{Z}/2\mathbb{Z}$-homology is a map of Hopf algebras.*

Proof: Let $\iota : S^4 \to BSU(2)$ be the standard inclusion and $E : S^4 \to \Omega S^5$ the suspension map. Now consider the following diagram which is commutative by naturality of the constructions involved. All horizontal maps are given by composition with f.

$$\begin{tikzcd}
\mathrm{Maps}(X, \Omega S^5) \arrow[r, "\circ f"] & \Omega^4 S^5 \\
\mathrm{Maps}(X, S^4) \arrow[r, "\circ f"] \arrow[u, "E\circ"'] \arrow[d, "\iota\circ"] & \Omega^3 S^4 \arrow[u, "E\circ"'] \arrow[d, "\iota\circ"] \\
\mathrm{Maps}(X, BSU(2)) \arrow[r, "f^\sharp"] & \Omega^2 S^3
\end{tikzcd}$$

All vertical arrows represent maps of H-spaces, and so does the top horizontal arrow. Notice that we are using here two different H-space structures on $\mathrm{Maps}(X, \Omega S^5)$ which agree up to homotopy, namely the one induced by the co-H-space structure on X and the one coming from the H-space structure of ΩS^5. Further notice that the maps given by composition with ι have a homotopy right-inverse, induced by the composition $\Omega BSU(2) \simeq S^3 \hookrightarrow \Omega S^4$ (which in the case of $\mathrm{Maps}(X, \cdot\,)$ depends on the choice of a desuspension of X). Hence they are surjective in homology. It follows from corollary 5.12 that the top two vertical arrows are injective in $\mathbf{Z}/2\mathbf{Z}$-homology. Since the composition $\mathrm{Maps}(X, S^4) \to \Omega^4 S^5$ as a map of H-spaces induces a map of Hopf algebras in homology and the top right vertical map is injective in $\mathbf{Z}/2\mathbf{Z}$-homology, it follows that the middle horizontal arrow is a map of Hopf algebras in $\mathbf{Z}/2\mathbf{Z}$-homology. This implies that the map $\mathrm{Maps}(X, S^4) \to \Omega^2 S^3$ is a map of Hopf algebras in $\mathbf{Z}/2\mathbf{Z}$-homology, and since the bottom left vertical arrow is surjective in homology and preserves the Hopf algebra structure, it follows that $f^\sharp_*$ in $\mathbf{Z}/2\mathbf{Z}$-homology is a map of Hopf algebras. $\square$

Recall that $H_*(\mathrm{Maps}(X, BSU(2)); \mathbf{Z}/2\mathbf{Z})$ is the symmetric Hopf algebra primitively generated by $H^2(X; \mathbf{Z}/2\mathbf{Z})$. Lemma 4.2 and theorem 5.9 imply that $H_*(\Omega^2\mathcal{F}; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}[c_1^2, c_2, c_3, \ldots]$ where the generators $c_i \in H_{2i}(\Omega^2 S^3)$ are primitive.

Corollary 5.14 *The map*

$$f_*^{\sharp}: \mathcal{S}^*(H^2(X;\mathbf{Z}/2\mathbf{Z})) \longrightarrow \mathbf{Z}/2\mathbf{Z}[c_1^2, c_2, c_3, \ldots]$$

is the map of Hopf algebras given on generators by $\alpha \mapsto Q_f(\alpha,\alpha)\cdot c_1^2$ *for* $\alpha \in H^2(X)$.

Proof: In view of theorem 5.13 all we have to do is determine the map in degree 2. We know the map on second homotopy groups from theorem 4.5. The result now follows from the Hurewicz theorem. □

Let $H^2_{\text{ev}}(Y) = \{\alpha \in H^2(Y) \mid Q_f(\alpha,\alpha) \text{ is even}\}$ and $H^2_{\text{ev}}(Y;\mathbf{Z}/2\mathbf{Z})$ the image of $H^2_{\text{ev}}(Y)$ in $H^2(Y;\mathbf{Z}/2\mathbf{Z})$. Let

$$\mathcal{I}_Y = \begin{cases} \{(i,j)\in\mathbf{Z}^2 \mid i\geq 2, j\geq 0\} & \text{if } Q_f \text{ is odd} \\ \{(i,j)\in\mathbf{Z}^2 \mid i\geq 1, j\geq 0, (i,j)\neq(1,0)\} & \text{if } Q_f \text{ is even}\end{cases}$$

Corollary 5.15 *Taking coefficients in* $k=\mathbf{Z}/2\mathbf{Z}$, *there is a natural isomorphism of coalgebras*

$$\mathrm{Cotor}^{H_*(\Omega^2\mathcal{F})}\Big(H_*\big(\mathrm{Maps}(X,BSU(2))\big),k\Big) \cong \mathcal{S}^*(H^2_{\text{ev}}(Y;k))\otimes \bigotimes_{(i,j)\in\mathcal{I}_Y} k[y_{i,j}]$$

where the right hand side is primitively generated as a Hopf algebra and $\deg y_{i,j} = 2^j(2^i-1)-1$.

Proof: Suppose the intersection form Q_f of Y is odd. Choose $b\in H^2(Y;k)$ such that $Q_f(b,b)$ is odd. Then, as Hopf algebras,

$$\mathcal{S}^*(H^2(X;k)) \cong \mathcal{S}^*(H^2_{\text{ev}}(Y;k))\otimes k[b]$$

where the co-action of $H_*(\Omega^2\mathcal{F})$ on the first factor is trivial. Hence, as coalgebras, $\mathrm{Cotor}^{H_*(\Omega^2\mathcal{F})}\Big(H_*\big(\mathrm{Maps}(X,BSU(2))\big),k\Big)$ is isomorphic to

$$\mathrm{Cotor}^k\Big(\mathcal{S}^*(H^2_{\text{ev}}(Y,k)),k\Big)\otimes\mathrm{Cotor}^{k[c_1^2]}\Big(k[b],k\Big)\otimes\mathrm{Cotor}^{k[c_2,c_3,\ldots]}\Big(k,k\Big).$$

Now, $\mathrm{Cotor}^{k[c_1^2]}(k[b],k) \cong k$ and the result becomes independent of the choice of b. Notice that $\mathrm{Cotor}^{k}(\mathcal{S}^*(H^2_{\mathrm{ev}}(Y;k)), k) \cong \mathcal{S}^*(H^2_{\mathrm{ev}}(Y;k))$. There is a decomposition of coalgebras

$$\mathbf{Z}/2\mathbf{Z}[c_2, c_3, \ldots] \cong \bigotimes_{i,j} \Lambda\{c_i^{(2^j)}\},$$

$\mathrm{Cotor}^{\Lambda\{c_i^{(2^j)}\}}(k,k) \cong \mathbf{Z}/2\mathbf{Z}[y_{i,j}]$ and $\deg y_{i,j} = \deg c_i^{(2^j)} - 1 = 2^j(2^i - 1) - 1$.

Now suppose that Q_f is even. $\mathcal{S}^*(H^2(Y;k)) = \mathcal{S}^*(H^2_{\mathrm{ev}}(Y;k))$ and the co-action of $H_*(\Omega^2\mathcal{F})$ is trivial, so, as coalgebras,

$$\begin{aligned}
\mathrm{Cotor}^{H_*(\Omega^2\mathcal{F})}\big(\mathcal{S}^*(H^2(Y;k)),\big) &\cong \mathcal{S}^*(H^2(Y;k)) \otimes \mathrm{Cotor}^{k[c_1^2, c_2, c_3, \ldots]}\big(k, k\big) \\
&\cong \mathcal{S}^*(H^2(Y;k)) \otimes \bigotimes_{(i,j)\in\mathcal{I}_Y} \mathrm{Cotor}^{\Lambda\{c_i^{(2^j)}\}}\big(k, k\big) \\
&\cong \mathcal{S}^*(H^2(Y;k)) \otimes \bigotimes_{(i,j)\in\mathcal{I}_Y} k[y_{i,j}]
\end{aligned}$$

$\square$

For our further calculations we need the following lemma.

Lemma 5.16 *Let Y have even intersection form.*

i) The map ι_0: $\mathrm{Maps}(Y, S^4) \to \mathrm{Maps}(Y, BSU(2))$ induces a surjection in $\mathbf{Z}/2\mathbf{Z}$-homology.

ii) The Leray-Serre spectral sequence of the fibration

$$\Omega_0^4 S^3 \longrightarrow \mathrm{Maps}(Y, BSU(2)) \longrightarrow \mathrm{Maps}(X, BSU(2))$$

in $\mathbf{Z}/2\mathbf{Z}$-homology collapses at E^2.

The second statement was shown by G. Masbaum [19] using Bott periodicity. We give an independent proof.

Proof: Consider the following diagram, which extends the diagram in the proof of theorem 5.13. The horizontal arrows represent sequences of (homotopy) fibrations. The top vertical maps are induced by the map $E\colon S^4 \to \Omega S^5$. They are injective in $\mathbb{Z}/2\mathbb{Z}$-homology according to theorem 5.12, with the possible exception of g. As in the proof of theorem 5.13 one shows that the bottom vertical maps are surjective in homology, with the possible exception of h.

$$\begin{tikzcd}
\Omega^5_0 S^5 \arrow[r] & \mathrm{Maps}_0(Y, \Omega S^5) \arrow[r] & \mathrm{Maps}(\Sigma X, S^5) \arrow[r, "\circ\Sigma f"] & \Omega^4 S^5_{<5>} \\
\Omega^4_0 S^4 \arrow[r] \arrow[u] \arrow[d] & \mathrm{Maps}_0(Y, S^4) \arrow[r] \arrow[u, "g"] \arrow[d, "h"] & \mathrm{Maps}(X, S^4) \arrow[r, "\circ f"] \arrow[u] \arrow[d] & \Omega^3 S^4_{<4>} \arrow[u] \arrow[d] \\
\Omega^3_0 S^3 \arrow[r] & \mathrm{Maps}_0(Y, BSU(2)) \arrow[r] & \mathrm{Maps}(X, BSU(2)) \arrow[r, "f^1"] & \Omega^2 \mathcal{F}
\end{tikzcd}$$

Since Y has even intersection form, Σf is null-homotopic (lemma 2.6). It follows that the top right horizontal map in the diagram is null-homotopic, so $\mathrm{Maps}_0(Y, \Omega S^5)$ has the homotopy type of the product $\Omega^5_0 S^5 \times \mathrm{Maps}(\Sigma X, S^5)$ and the Leray-Serre spectral sequence of the fibration

$$\Omega^5_0 S^5 \longrightarrow \mathrm{Maps}_0(Y, \Omega S^5) \longrightarrow \mathrm{Maps}(\Sigma X, S^5)$$

collapses at E^2. But the spectral sequence of the fibration

$$\Omega^4_0 S^4 \longrightarrow \mathrm{Maps}_0(Y, S^4) \longrightarrow \mathrm{Maps}(X, S^4)$$

maps injectively into the top spectral sequence, so it collapses, too. The spectral sequence of the middle row maps surjectively to the bottom fibration

$$\Omega^3_0 S^3 \longrightarrow \mathrm{Maps}_0(Y, BSU(2)) \longrightarrow \mathrm{Maps}(X, BSU(2))$$

so this spectral sequence collapses as well, and the map h is surjective in homology. $\square$

Theorem 5.17 *For coefficients in $k = \mathbb{Z}/2\mathbb{Z}$, the Eilenberg-Moore spectral sequence of the fibration*

$$\mathrm{Maps}(Y, BSU(2)) \longrightarrow \mathrm{Maps}(X, BSU(2)) \xrightarrow{f^2} \Omega^2\mathcal{F}$$

collapses at E^2.

Proof: For Y with even intersection form it follows from corollaries 5.11 and 5.15 that the E^2-term of the Eilenberg-Moore spectral sequence is isomorphic to the E^2-term of the Leray-Serre spectral sequence of the fibration of lemma 5.16, which collapses at E^2, hence the result.

Now let $Y \simeq \mathbb{CP}^2$. It follows from corollary 5.14 that η^2 is injective in $\mathbb{Z}/2\mathbb{Z}$-homology, so from the fibration

$$\mathrm{Maps}(\mathbb{CP}^2, BSU(2)) \xrightarrow{\rho} \mathrm{Maps}(S^2, BSU(2)) \xrightarrow{\eta^2} \Omega^2\mathcal{F}$$

we conclude that the map ρ_* is zero in homology. There is a spectral sequence with E^2-term

$$\mathrm{Cotor}^{H_*\left(\mathrm{Maps}(S^2, BSU(2))\right)}\left(H_*\left(\mathrm{Maps}(\mathbb{CP}^2, BSU(2))\right), k\right)$$

converging to the homology of the homotopy fibre of ρ, which is $\Omega^3_0 S^3$. Since ρ_* is zero, we get

$$\begin{aligned}
&\mathrm{Cotor}^{H_*\left(\mathrm{Maps}(S^2, BSU(2))\right)}\left(H_*(\mathrm{Maps}(\mathbb{CP}^2, BSU(2))), k\right) \\
&\quad\cong H_*\left(\mathrm{Maps}(\mathbb{CP}^2, BSU(2))\right) \otimes \mathrm{Cotor}^{H_*(\Omega S^3)}\left(k, k\right) \\
&\quad\cong H_*\left(\mathrm{Maps}(\mathbb{CP}^2, BSU(2))\right) \otimes \bigotimes_{m \geq 1} k[\, c_m \,]
\end{aligned}$$

where $\deg c_m = 2^m - 1$. With corollary 5.15, $H_*\left(\mathrm{Maps}(\mathbb{CP}^2, BSU(2))\right)$ is a subquotient of

$$\bigotimes_{(i,j) \in T_{\mathbb{CP}^2}} k[\, y_{i,j} \,]$$

where $i \geq 2$, $j \geq 0$ and $\deg y_{i,j} = 2^j(2^i - 1) - 1$. Notice that, if we write c_m as $y_{1,m}$, the formula for $\deg y_{i,j}$ gives the correct result for $\deg c_m$. Putting these results together, we get that $H_*(\Omega_0^3 S^3)$ is a subquotient of

$$
\begin{aligned}
\bigotimes_{(i,j)\in\mathcal{I}_{\mathbb{C}P^2}} k[\, y_{i,j} \,] \otimes \bigotimes_{m\geq 1} k[\, y_m \,] \quad &\cong \bigotimes_{(i,j)\in\mathcal{I}_{S^4}} k[\, y_{i,j} \,] \\
&\overset{5.15}{\cong} \operatorname{Cotor}^{H_*(\Omega^2\mathcal{F})}(k, k) \\
&\overset{5.11}{\cong} H_*(\Omega_0^3 S^3)
\end{aligned}
$$

It follows that all subquotients involved are actually isomorphisms, which means that both spectral sequences involved collapse at E^2. Notice that this implies that the map

$$H_*(\Omega_0^3 S^3; k) \longrightarrow H_*\big(\mathrm{Maps}(\mathbb{C}P^2, BSU(2)); k\big)$$

is surjective. Also, the same argument as for $\mathbb{C}P^2$ also applies to any complex $Y \simeq e^4 \cup_f S^2$ where f represents an odd multiple of the Hopf map.

Now let Y be arbitrary with odd intersection form. As in lemma 2.7, construct a map $p\colon S^2 \to X$ and let K and L be defined by the cofibrations $S^2 \xrightarrow{p} X \xrightarrow{q} K$ and $S^2 \xrightarrow{p} Y \to L$. Choose $t\colon K \to X$ to be a right homotopy inverse of q. Notice that the composition $S^2 \xrightarrow{p} X \to X/K$ is a homotopy equivalence, where X/K denotes the homotopy cofibre of t. From t we obtain a map $\zeta\colon K \to Y$ such that the following diagram of cofibrations is commutative, where C is the homotopy cofibre of ζ.

$$
\begin{tikzcd}
S^2 \arrow[r] & C \arrow[r] & S^4 \\
S^2 \arrow[u, equal] \arrow[r, "p"] & Y \arrow[u, "r"'] \arrow[r] & L \arrow[u] \\
* \arrow[u] \arrow[r] & K \arrow[u, "\zeta"'] \arrow[r, equal] & K \arrow[u]
\end{tikzcd}
$$

It now follows that $C \simeq e^4 \cup_g S^2$ for some $g \in \pi_3(S^2)$ and $r \circ p$ is a generator of $\pi_2(C)$. It follows from lemma 2.7 that L is a complex with even intersection form and $H^2(L; \mathbb{Z}/2\mathbb{Z}) \cong H^2_{ev}(Y; \mathbb{Z}/2\mathbb{Z})$. Let α be a generator of $H^2(C)$. We see from the following calculation that g is an odd multiple of the Hopf map.

$$\begin{aligned} Q_g(\alpha, \alpha) &= Q_f(r^*(\alpha), r^*(\alpha)) \\ &\overset{2.5}{\equiv} \left\langle r^*(\alpha), p_*([S^2]) \right\rangle \pmod 2 \\ &= \left\langle p^*(r^*(\alpha)), [S^2] \right\rangle \\ &= \pm 1 \end{aligned}$$

Now apply the functor $\mathrm{Maps}_0(\,\cdot\,, BSU(2))$ to the above diagram to obtain the following map of fibrations.

$$\begin{tikzcd} \mathrm{Maps}_0(S^4, BSU(2)) \arrow[r] \arrow[d, dash] & \mathrm{Maps}_0(L, BSU(2)) \arrow[r] \arrow[d, dash] & \mathrm{Maps}(K, BSU(2)) \arrow[d, equal] \\ \mathrm{Maps}_0(C, BSU(2)) \arrow[r] & \mathrm{Maps}_0(Y, BSU(2)) \arrow[r] & \mathrm{Maps}(K, BSU(2)) \end{tikzcd}$$

According to the previous part of the proof, the left vertical arrow induces a surjection in $\mathbb{Z}/2\mathbb{Z}$-homology. The Leray-Serre spectral sequence of the top fibration collapses at E^2 according to theorem 5.16, hence so does the Leray-Serre spectral sequence of the bottom fibration and we obtain

$$\begin{aligned} & H_*(\mathrm{Maps}_0(Y, BSU(2))) \\ &\qquad \cong H_*(\mathrm{Maps}(K, BSU(2))) \otimes H_*(\mathrm{Maps}_0(C, BSU(2))) \\ &\qquad \cong S^*(H^2_{ev}(Y; k)) \otimes \bigotimes_{(i,j) \in I_C} k[y_{i,j}] \\ &\qquad \overset{5.15}{\cong} \mathrm{Cotor}^{H_*(\Omega^2 \mathcal{F})}\Big(\mathrm{Maps}(X, BSU(2)), k\Big) \end{aligned}$$

and the Eilenberg-Moore spectral sequence collapses. □

Corollary 5.18 *The map* $\iota \circ$: $\mathrm{Maps}(Y, S^4) \to \mathrm{Maps}(Y, BSU(2))$ *induces a surjection in* $\mathbb{Z}/2\mathbb{Z}$*-homology.*

Proof: If Y has even intersection form, the result is just the statement of lemma 5.16*i*, so assume Y has odd intersection form. Then, with the constructions and notations of the proof of theorem 5.17, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Maps}(L, S^4) \arrow[r] \arrow[d] & \mathrm{Maps}(L, BSU(2)) \arrow[d] \\ \mathrm{Maps}(Y, S^4) \arrow[r] & \mathrm{Maps}(Y, BSU(2)) \end{tikzcd}$$

where the top horizontal and right hand vertical maps are surjective in $\mathbb{Z}/2\mathbb{Z}$-homology, hence so is the bottom horizontal map. □

Corollary 5.19 *There is a natural isomorphism of vector spaces*

$$H_*\Big(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/2\mathbb{Z}\Big) \cong \mathcal{S}^*(H^2_{\mathrm{ev}}(Y; \mathbb{Z}/2\mathbb{Z})) \otimes \bigotimes_{(i,j)\in\mathcal{I}_Y} \mathbb{Z}/2\mathbb{Z}[y_{i,j}]$$

Proof: Corollary 5.15 and theorem 5.17. □

Remark: Corollary 5.19 shows that, as a vector space, the $\mathbb{Z}/2\mathbb{Z}$-homology of $H_*(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/2\mathbb{Z})$ depends only on the rank and the type of the intersection form. We identified the E^2-term as a coalgebra and the Eilenberg-Moore spectral sequence converges to its target as a coalgebra, so we are able to obtain some information about the coalgebra structure of $H_*(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/2\mathbb{Z})$. In particular, the diagonal of the image of $H_*(\Omega^3_0 S^3; \mathbb{Z}/2\mathbb{Z})$ is as one would expect and, taking e. g. $Y = \mathbb{C}P^2$, the isomorphism of corollary 5.19 becomes an isomorphism of coalgebras. On the other hand, the diagonal of elements corresponding to $\mathbb{Z}/2\mathbb{Z}[H^2_{\mathrm{ev}}(Y; \mathbb{Z}/2\mathbb{Z})]$ is given by the spectral sequence only 'up to lower filtrations' and it possibly depends on data of the intersection form beyond those of rank and type.

5.4 The Z/pZ-homology p ≥ 5

Throughout this section let $p \geq 5$ be a prime and, as before, $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$. We compute the homology of $\mathrm{Maps}(Y, BSU(2))$ with coefficients in $k = \mathbb{Z}/p\mathbb{Z}$. The result, originally due to G. Masbaum [19], is a direct consequence of theorem 4.18. We need the following well-known fact.

Lemma 5.20 *The 12-th power map $\ell\colon \Omega^2 S^3 \to \Omega^2 S^3$ induces an isomorphism in $\mathbb{Z}/p\mathbb{Z}$-homology.*

Proof: Since S^3 is an H-space, ℓ is homotopic to $\Omega^2\chi$, where $\chi : S^3 \to S^3$ is a map of degree 12, which induces an isomorphism in $\mathbb{Z}/p\mathbb{Z}$-homology. The map of path fibrations covering χ on the base induces an isomorphism of Eilenberg-Moore spectral sequences and the map $\Omega\chi$ induces an isomorphism in $\mathbb{Z}/p\mathbb{Z}$-homology. By repeating the same argument once more, we obtain the result. □

Theorem 5.21 *As a coalgebra, $H_*(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/p\mathbb{Z})$ is isomorphic to*

$$H_*\big(\mathrm{Maps}(X, BSU(2)); \mathbb{Z}/p\mathbb{Z}\big) \otimes H_*(\Omega_0^3 S^3; \mathbb{Z}/p\mathbb{Z})$$

Proof: It follows from lemma 5.20 that there is a $\mathbb{Z}/p\mathbb{Z}$-homology equivalence between the homotopy fibres of $f^\sharp$ and $12f^\sharp$. But, according to theorem 4.18, $12f^\sharp$ is contractible, so the homotopy fibre is a product and the result follows from the Künneth formula. □

Remark: This result can be formulated as saying that, for $p \geq 5$, there is a p-local equivalence (see [2])

$$\mathrm{Maps}(Y, BSU(2)) \cong \Omega_0^3 S^3 \times \mathrm{Maps}(X, BSU(2)).$$

Explicitly, the homology can now be obtained using theorems 5.10*ii* and 4.10.

Corollary 5.22 *As a coalgebra, $H_*(\mathrm{Maps}(Y, BSU(2)); \mathbf{Z}/p\mathbf{Z})$ is isomorphic to*

$$\bigotimes_{i>0} k[z_i] \otimes \bigotimes_{i,j\geq 0} \Lambda\{u_{i,j}\} \otimes \bigotimes_{i,j\geq 0} k[y_{i,j}] \otimes \mathcal{S}^*(H^2(X; \mathbf{Z}/p\mathbf{Z}))$$

where $\deg z_i = 2p^i - 2$, $\deg u_{i,j} = p^j(2p^{i+1} - 2) - 1$ and $\deg y_{i,j} = p^{j+1}(2p^{i+1} - 2) - 2$ and the diagonal is given by the diagonal of $\Omega_0^3 S^3$ and the fact that $\mathcal{S}^*(H^2(X))$ is primitively generated. □

Corollary 5.23 *In $\mathbf{Z}/p\mathbf{Z}$-homology, the Eilenberg-Moore spectral sequence of the fibration*

$$\mathrm{Maps}_0(Y, BSU(2)) \longrightarrow \mathrm{Maps}_0(X, BSU(2)) \longrightarrow \Omega^2\mathcal{F}$$

collapses at E^2. □

5.5 The Z/3Z-homology

The homology of $\mathrm{Maps}(Y, BSU(2))$ with coefficients in $\mathbf{Z}/3\mathbf{Z}$ differs essentially from the results at other primes. For primes greater than 3, the homology only depends on the rank of the intersection form of Y, and for the prime 2 it is (multiplicatively up to higher filtrations) determined by the rank and the type of the intersection form. The main result of this section is to show that the mod 3 intersection form of Y can be completely recovered from the $\mathbf{Z}/3\mathbf{Z}$-homology and cohomology of $\mathrm{Maps}(Y, BSU(2))$. We start by determining the map induced by $f^\sharp$ in $\mathbf{Z}/3\mathbf{Z}$-homology. Recall from theorem 4.10 that $H_*(\mathrm{Maps}(X, BSU(2))) \cong \mathcal{S}^*(H^2(X))$. Its dual $\mathrm{Hom}(H_*(\mathrm{Maps}(X, BSU(2)), \mathbf{Z}) \cong \Gamma_*(H_2(X))$ contains the divided power algebra $\Gamma_*(Q)$ where $Q \in \Gamma_2(H_2(X))$ is the intersection form of Y. Let $e \in H_4(\Omega^2\mathcal{F}; \mathbf{Z}/3\mathbf{Z}) \cong \mathbf{Z}/3\mathbf{Z}$ be the standard generator obtained via the mod 3 Hurewicz map. Recall from theorem 5.10 that $H_*(\Omega^2\mathcal{F}; \mathbf{Z}/3\mathbf{Z})$ contains the polynomial algebra $\mathbf{Z}/3\mathbf{Z}[e]$ as a Hopf subalgebra.

Theorem 5.24 *The map*

$$f_*^\sharp\colon H_*(\mathrm{Maps}(X, BSU(2)); \mathbf{Z}/3\mathbf{Z}) \to H_*(\Omega^2\mathcal{F}; \mathbf{Z}/3\mathbf{Z}).$$

is given by

$$\alpha_1\cdots\alpha_{2n} \longmapsto \frac{1}{n!}Q^n(\alpha_1\cdots\alpha_{2n})\cdot c^n$$

for $\alpha_1,\ldots,\alpha_{2n} \in H^2(X)$.

Remark: Explicitly, the maps Q^n can be computed via the formula

$$Q^n(\alpha_1\cdots\alpha_{2n}) = \frac{1}{2^n}\sum_{\sigma\in\Sigma_{2n}} Q(\alpha_{\sigma(1)},\alpha_{\sigma(2)})\cdots Q(\alpha_{\sigma(2n-1)},\alpha_{\sigma(2n)})$$

Proof: Since we are concerned with $\mathbf{Z}/3\mathbf{Z}$-homology, $f^\sharp$ and $4\cdot f^\sharp$ induce the same map, so we can without loss of generality assume that the intersection form Q is even. According to theorem 4.15, the map $f^\sharp$ factors as

$$\mathrm{Maps}(X, BSU(2)) \xrightarrow{\gamma(\tilde{f})} \Omega^2 S^6 \xrightarrow{\Omega^2\rho} \Omega^2\mathcal{F}$$

where $\tilde{f}$ is in the inverse image of the element $f\in\pi_3(X)$ under the composition $\pi_2(X)\otimes\pi_2(X) \to \mathcal{S}^2(\pi_2(X)) \to \pi_3(X)$ and $\rho\colon S^6 \to S^3$ is the standard generator of $\pi_6(S^3)$. Let $\tilde{c}$ be the standard generator of $H_4(\Omega^2 S^6;\mathbf{Z})$ and $c^* \in H^4(\Omega^2 S^6)$ its dual. It follows that $\Omega^2\rho$ maps $\tilde{c}$ mod 3 to c, so, for $\alpha,\beta\in H^2(X)$,

$$f_*^\sharp(\alpha\cdot\beta) = \left(\left\langle c^*, \gamma(\tilde{f})_*(\alpha\cdot\beta)\right\rangle \bmod 3\right)\cdot c.$$

Furthermore, using the map θ_3 of theorem 4.17, we obtain

$$\begin{aligned}\left\langle c^*, \gamma(\tilde{f})_*(\alpha\cdot\beta)\right\rangle &= \left\langle \theta_3(\gamma(\tilde{f})), \alpha\cdot\beta\right\rangle \\ &\overset{4.17}{=} Q(\alpha,\beta)\end{aligned}$$

so in dimension four the map is as stated. Next consider the following diagram, where the dashed arrows represent maps of sets and the unbroken arrows group homomorphisms.

$$\begin{tikzcd}
\pi_2(X) \times \pi_2(X) \arrow[r, dashed] \arrow[d, dashed] & {[\mathrm{Maps}(X, BSU(2)), \Omega S^5]} \arrow[d, "\Omega E\circ"] \\
\pi_2(X) \otimes \pi_2(X) \arrow[r, "\gamma"] & {[\mathrm{Maps}(X, BSU(2)), \Omega^2 S^6]}
\end{tikzcd}$$

In order to describe the top horizontal arrow and see that the diagram commutes, recall from the proof of theorem 4.17 that the map $\Omega S^3 \times \Omega S^3 \to \Omega^2 S^6$ which sends (f_1, f_2) to $f_1 \wedge f_2$ factors through $\Omega S^5 \xrightarrow{\Omega E} \Omega^2 S^6$. It follows that the composition of the left and bottom arrows in above diagram, which represents the map $\wedge^\sharp \circ \Big((i \circ g) \times (i \circ g)\Big)$ in the proof of theorem 4.15, factors through

$$[\mathrm{Maps}(X, BSU(2)), \Omega S^5] \xrightarrow{\Omega E\circ} [\mathrm{Maps}(X, BSU(2)), \Omega^2 S^6]$$

and the top horizontal map is now chosen to be the one arising from this factorization. Since $\pi_2(X) \times \pi_2(X)$ generates $\pi_2(X) \otimes \pi_2(X)$ as a group, and since the right hand vertical arrow in above diagram is a group homomorphism, it follows that there is a map τ such that the map $\gamma(\hat{f})$ factors (non-naturally) as

$$\mathrm{Maps}(X, BSU(2)) \xrightarrow{\tau} \Omega S^5 \xrightarrow{\Omega E} \Omega^2 S^6.$$

According to theorem 4.9, $H_*(\Omega S^5) \cong \mathbf{Z}[\hat{c}]$, and in $\mathbf{Z}/3\mathbf{Z}$-homology this maps to the factor $\mathbf{Z}/3\mathbf{Z}[c] \subset H_*(\Omega^2 \mathcal{F})$, since the map $\Omega^2\rho \circ \Omega E \colon \Omega S^5 \to \Omega^2 \mathcal{F}$ is a loop map. Let $c^* \in H^4(\Omega S^5)$ denote the dual of $\hat{c}$. The dual of $(\hat{c})^n \in H_{4n}(\Omega S^5)$ is $\frac{1}{n!}(c^*)^n$ and

$$\begin{aligned}
f_*^\sharp(\alpha_1 \cdots \alpha_{2n}) &= \left\langle \frac{1}{n!}(c^*)^n, \tau_*(\alpha_1 \cdots \alpha_{2n}) \right\rangle \cdot c^n \\
&= \left\langle \frac{1}{n!} Q^n, \alpha_1 \cdots \alpha_{2n} \right\rangle \cdot c^n \\
&= \frac{1}{n!} Q^n(\alpha_1 \cdots \alpha_{2n}) \cdot c^n.
\end{aligned}$$

In order to see the explicit formula given in the remark, recall that

$$\begin{aligned} Q^n(\alpha_1\cdots\alpha_{2n}) &= Q\otimes Q^{n-1}\Big(\Delta(\alpha_1\cdots\alpha_{2n})\Big) \\ &= \frac{1}{2}\sum_{i\neq j}\left(Q(\alpha_i,\alpha_j)\cdot Q^{n-1}(\prod_{i\neq k\neq j}\alpha_k)\right) \end{aligned}$$

and the result follows by induction. □

With this information one could now, in principle, compute

$$\mathrm{Cotor}^{H_*(\Omega^2\mathcal{F};\mathbf{Z}/3\mathbf{Z})}\Big(H_*\big(\mathrm{Maps}(X,BSU(2));\mathbf{Z}/3\mathbf{Z}\big)\,,\,\mathbf{Z}/3\mathbf{Z}\Big)$$

and thus obtain information about the $\mathbf{Z}/3\mathbf{Z}$ homology of $\mathrm{Maps}(Y,BSU(2))$. We have at present not yet succeeded in carrying out this computation in general and we do not know whether the Eilenberg-Moore spectral sequence collapses for this case. Instead, we show that the mod 3 intersection form of Y can be recovered from the $\mathbf{Z}/3\mathbf{Z}$ cohomology of the mapping space. In the following let $Q\in\Gamma_2(H^2(X))$ be the intersection form of $Y\simeq e^4\cup_f S^2\vee\ldots\vee S^2$, $\overline{Q}$ the mod 3 reduction of Q and $r\colon\ \mathrm{Maps}(Y,BSU(2))\to\mathrm{Maps}(X,BSU(2))$ the restriction map.

Lemma 5.25

i) *The map r induces an isomorphism*

$$H^2(\mathrm{Maps}(X,BSU(2));\mathbf{Z}/3\mathbf{Z})\xrightarrow{\cong}H^2(\mathrm{Maps}(Y,BSU(2));\mathbf{Z}/3\mathbf{Z})$$

ii) *In $\mathbf{Z}/3\mathbf{Z}$ cohomology, there is an exact sequence*

$$H^4(\Omega^2\mathcal{F})\xrightarrow{(f^\flat)^*}H^4(\mathrm{Maps}(X,BSU(2)))\xrightarrow{r^*}H^4(\mathrm{Maps}(Y,BSU(2))).$$

Proof: Consider the Leray-Serre spectral sequence of the fibration

$$\Omega^3_0 S^3\longrightarrow\mathrm{Maps}_0(Y,BSU(2))\longrightarrow\mathrm{Maps}(X,BSU(2))$$

in $\mathbb{Z}/3\mathbb{Z}$-cohomology. The lowest non-trivial cohomology of the fibre is $\mathbb{Z}/3\mathbb{Z}$ in dimension three, so the first statement follows immediately. By comparing above spectral sequence with the one of the path fibration over $\Omega^2\mathcal{F}$, one sees that the differential $d_4\colon E_4^{0,3} \to E_4^{4,0}$ has the same image as the map $(f^*)^*$ in dimension four, and the second statement follows. □

Now let $\mathcal{J}^* = im\ r^* \subseteq H^*(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/3\mathbb{Z})$.

Corollary 5.26

i) There is a natural isomorphisms

$$H^2\Big(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/3\mathbb{Z}\Big) \cong H_2(X) \otimes \mathbb{Z}/3\mathbb{Z}.$$

ii) There is a natural isomorphism

$$\mathcal{J}^4 \cong \left(\Gamma_2(H_2(X))\Big/\langle Q\rangle\right) \otimes \mathbb{Z}/3\mathbb{Z}.$$

iii) The cup product pairing

$$\mathcal{S}^2\Big(H^2\Big(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/3\mathbb{Z}\Big)\Big) \longrightarrow H^4\Big(\mathrm{Maps}(Y, BSU(2))\mathbb{Z}/3\mathbb{Z}\Big)$$

is given by the reduction of the natural map $\mathcal{S}^2(H_2(X)) \to \Gamma_2(H_2(X))$ modulo three, followed by projection.

Proof: The first two statements follow immediately from lemma 5.25 and the dual statement of theorem 5.24. In order to verify the third statement, notice that the cup product on $H^2\big(\mathrm{Maps}(Y, BSU(2)); \mathbb{Z}/3\mathbb{Z}\big)$ is completely determined by the product on $H^2\big(\mathrm{Maps}(X, BSU(2)); \mathbb{Z}/3\mathbb{Z}\big)$, where the corresponding statement is obvious. □

As a consequence, the mod 3 intersection form of Y can be recovered from the homology and cohomology as follows. The cup product pairing determines a map

$$\kappa : \Gamma_2\Big(H^2\big(\mathrm{Maps}(Y, BSU(2)); \mathbf{Z}/3\mathbf{Z}\big)\Big) \to H^4\big(\mathrm{Maps}(Y, BSU(2)); \mathbf{Z}/3\mathbf{Z}\big)$$

and according to corollary 5.26, $ker\ \kappa = \left\langle \overline{Q} \right\rangle$ is at most one-dimensional. Let $\Theta \in \Gamma_2\Big(H^2\big(\mathrm{Maps}(Y, BSU(2)); \mathbf{Z}/3\mathbf{Z}\big)\Big)$ be defined to be zero if $ker\ \kappa = 0$ and to be a generator of $ker\ \kappa$ otherwise. The Kronecker pairing with Θ determines a map

$$\theta : S^2\Big(H_2\big(\mathrm{Maps}(Y, BSU(2)); \mathbf{Z}/3\mathbf{Z}\big)\Big) \to \mathbf{Z}/3\mathbf{Z}.$$

Since $\Theta = \pm\overline{Q}$, this pairing is, under the natural identification

$$H_2\big(\mathrm{Maps}(Y, BSU(2)); \mathbf{Z}/3\mathbf{Z}\big) \cong H^2(Y; \mathbf{Z}/3\mathbf{Z}),$$

just the mod 3 intersection form of Y, where the choice of Θ reflects the choice of the orientation of Y. Thus we have proved the following.

Theorem 5.27 *Let* $Y \simeq e^4 \cup_f S^2 \vee \ldots \vee S^2$, $Z \simeq e^4 \cup_g S^2 \vee \ldots \vee S^2$ *and suppose that* $\mathrm{Maps}(Y, BSU(2)) \simeq \mathrm{Maps}(Z, BSU(2))$. *Then, up to orientation, the* mod 3 *intersection forms of* Y *and* Z *are isomorphic.*

Corollary 5.28 *Let* M, N *be closed, simply-connected four-manifolds with intersection forms* Q_M *and* Q_N. *If* $\mathrm{Maps}(M, BSU(2)) \simeq \mathrm{Maps}(N, BSU(2))$, *then* $\sigma(Q_M) \equiv \pm\sigma(Q_N) \pmod 4$.

Proof: Since $\mathrm{rank}\, Q_M = \dim H^2\big(\mathrm{Maps}(M, BSU(2)); \mathbf{Z}/3\mathbf{Z}\big)$, we know that $\mathrm{rank}\, Q_M = \mathrm{rank}\, Q_N$, so $\sigma(Q_M) \equiv \sigma(Q_N) \pmod 2$. If $\sigma(Q_M)$ and $\sigma(Q_N)$ are both odd, necessarily either their sum or their difference is divisible by four, so the assertion is proved. Suppose that $\sigma(Q_M)$ and $\sigma(Q_N)$ are both even.

In this case $\det Q_M$ is unchanged when reversing the orientation, so using theorem 5.27 and the fact that the determinant of the intersection form is ± 1, we get that $\det Q_M = \det Q_N$. It follows from lemmas 3.16 and 3.13 that the mod 4 reduction of the signature is determined by the rank and the determinant, so $\sigma(Q_M) \equiv \sigma(Q_N) \pmod 4$. (Notice that in this case, $\sigma(Q_M) \equiv -\sigma(Q_M) \pmod 4$.) □

5.6 The Classification of Homotopy Types

As in section 4.1, let M be a smooth, closed, simply-connected four-manifold and let P be a principal $SU(2)$ bundle over M. Since the space $\mathcal{A}/\mathcal{G}^\bullet$ of connections on P modulo based gauge equivalence has the weak homotopy type of $\mathrm{Maps}_0(M, BSU(2))$, we can now summarize, how much information about the topology of M is contained in the topology of $\mathcal{A}/\mathcal{G}^\bullet$. Recall from Freedman's theorem that a smooth four-manifold is, up to oriented homeomorphism, determined by its intersection form, which, in turn, is determined by its rank, type and signature. This follows from theorem 3.4 for indefinite forms and for definite forms from Donaldson's theorem (see theorem 3.6).

First of all, it is clear from section 4.2 that the information about the orientation of M is lost as well as any information about the isomorphism class of the bundle P when considering the homotopy type of the space $\mathcal{A}/\mathcal{G}^\bullet$. That means that the remaining data of the manifold one could hope to recover are the rank and type of the intersection forms as well as the signature up to sign.

According to corollary 4.6, the type of the intersection form can be recovered from $\pi_1(\mathcal{A}/\mathcal{G}^\bullet)$, which is non trivial if and only if the intersection form is of even type, i. e. if and only if the manifold is spin. Also from section 4.2 it follows that the rank of the intersection form is equal to the dimension of the vector space $\pi_2(\mathcal{A}/\mathcal{G}^\bullet) \otimes \mathbf{Q}$, so type and rank can be recovered from the homotopy groups of $\mathcal{A}/\mathcal{G}^\bullet$. Theorems 4.4 and 4.7 assert that the homotopy groups of $\mathcal{A}/\mathcal{G}^\bullet$ are determined by the rank and type of the intersection form, so no further information can be drawn from the homotopy groups as such. It is, however, possible that further information is contained in homotopy operations. For example, composition with the Hopf map and Whitehead products give homotopy operations from $\pi_2(\mathcal{A}/\mathcal{G}^\bullet)$ to $\pi_3(\mathcal{A}/\mathcal{G}^\bullet)$, which can be used to define a natural map

$$S^2(\pi_2(\mathcal{A}/\mathcal{G}^\bullet)) \longrightarrow \mathbf{Z}/6\mathbf{Z}$$

but so far we have not succeeded in drawing any information about the intersection form from this other than what is also accessible via homological information.

The calculations in homology, in particular the case of coefficients in $\mathbf{Z}/3\mathbf{Z}$, make it possible to obtain some information about the signature (up to sign) of the intersection form of M. Corollary 5.28 asserts that the signature modulo *four* of the intersection form can be recovered (up to sign) from the cup product in $H^*(\mathcal{A}/\mathcal{G}^\bullet; \mathbf{Z}/3\mathbf{Z})$. On the other hand, corollary 4.12 sets a limit to what we could hope for: The homotopy type of $\mathcal{A}/\mathcal{G}^\bullet$ is in fact determined by the rank, type and the signature modulo *eight* (up to sign) of the intersection form, so this is all that could possibly be recovered.

For spin manifolds this means that we have determined completely all homotopy types the space $\mathcal{A}/\mathcal{G}^\bullet$ can attain. Since the signature of a spin manifold is always divisible by eight, the homotopy type of $\mathcal{A}/\mathcal{G}^\bullet$ depends only on the rank of the intersection form (corollary 4.13), which is necessarily even for spin manifolds. On the other hand, a spin manifold of rank $2n$ is realized by the n-fold connected sum $(S^2 \times S^2)\# \ldots \#(S^2 \times S^2)$, so for every even rank there is precisely one homotopy type of spaces of connections modulo based gauge equivalence on spin manifolds. On the other hand, there exists more than one homeomorphism type of smooth spin manifolds for some ranks. The K3 surface (see [12]), for example, has intersection form $2(-E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so the associated space of connections modulo based gauge equivalence has the same homotopy type as that associated to the connected sum of 11 copies of $S^2 \times S^2$.

For smooth manifolds with odd intersection form, we know from theorems 3.4 and 3.6 that the intersection form is isomorphic to $m(1) \oplus n(-1)$ for some $m, n \in \mathbb{N}$, and it follows that each such manifold is homeomorphic to a connected sum of m copies of $\mathbb{C}P^2$ and n copies of $\overline{\mathbb{C}P^2}$. Notice that reversing the orientation interchanges m and n. Again, corollary 4.12 states that the homotopy type of the associated space of connections modulo based gauge equivalence only depends on the rank of the intersection form and its signature (up to sign) modulo eight, i. e. on the values $m + n$ and $\pm(m - n)$ mod 8. For a given even rank this means that there are at most three different homotopy types, namely those corresponding to the signatures 0, ± 2 and 4 modulo eight. (Of course, for the existence of a form with signature 4 we have to assume a rank of at least four.) Corollary 5.28 enables us to distinguish the homotopy type corresponding to $\sigma \equiv \pm 2 \pmod 8$ from the other two types, but so far we have not been able to decide whether the homotopy types corresponding to $\sigma \equiv 0 \pmod 8$ and $\sigma \equiv 4 \pmod 8$ are the same. Similarly, for a given odd rank, there are at most two different homotopy types, corresponding to the signatures ± 1 and ± 3 modulo eight. Again, our methods do not suffice to decide whether these homotopy types are the same or different.

Remark: The missing piece of information from which we would be able to distinguish the yet undistinguished homotopy types is precisely the information about the isomorphism class of the mod four intersection form of the manifold. For this it would in fact be sufficient to know the associated mod four valued quadratic form (see section 3.3). Hence one might expect to obtain some information from the $\mathbb{Z}/4\mathbb{Z}$ homology of $\mathcal{A}/\mathcal{G}^{\bullet}$. A computation of this homology in low dimensions, however, has not so far given us any new information.

Chapter 6

Approximation by Configuration Spaces

6.1 Fibre Bundles and Spaces of Sections

In the following we introduce some notations about fibre bundles and spaces of sections and prove some technical results needed later. For a general reference about fibre bundles see [17].

Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibre bundle. A ***section*** of the bundle is a map $\sigma\colon B \to E$ such that $p \circ \sigma = \mathrm{id}|_B$. Let $\Gamma(B, E)$ be the set of sections of the bundle, topologized as a subspace of the space of all maps from B to E with the compact-open topology. If $B_0 \subset B$ and $F_0 \hookrightarrow E_0 \to B_0$ is a subbundle of $F \to E \to B$, let $\Gamma((B, B_0), (E, E_0))$ denote the subspace of $\Gamma(B, E)$ consisting of those sections which, restricted to B_0, are sections of the subbundle. If B has a basepoint $*$ and the fibre $E\big|_*$ over $*$ has a basepoint $\star$, we write $\Gamma^\bullet(B, E)$ for $\Gamma((B, \{*\}), (E, \{\star\}))$. Now suppose that for all $b \in B$ the fibre $F_b = E\big|_b$ has a basepoint ∞_b. We define the ***support*** of a section σ as the closure of the set $\{b \in B \mid \sigma(b) \neq \infty_b\}$. Let $\Gamma^c(B, E)$ denote the subspace of $\Gamma(B, E)$ consisting of all sections with compact support.

By a ***vector bundle*** over a space B we always mean a real orientable vector bundle and a frame is always meant to preserve the orientation. This is, however, not a serious restriction and all the following theorems hold similarly for non-orientable vector bundles. Let V be a rank-n vector bundle over B with a Riemannian metric, and let FV denote the bundle of orientable orthonormal frames of the fibres of V. Notice that the action of the special orthogonal group $SO(n)$ on $\mathbb{R}^n$ extends to an action on $S^n = \mathbb{R}^n \cup \infty$, leaving the two 'poles' 0 and ∞ fixed. In this way V gives rise to a sphere bundle V^+, adding to each fibre $V\big|_b$ a point at infinity, denoted by ∞_b. In the following it is always understood that ∞_b is the basepoint of the fibre $V\big|_b^+$.

For two bundles $F_1 \hookrightarrow E_1 \to B$ and $F_2 \hookrightarrow E_2 \to B$, the fibre product $E_1 \times_B E_2$ defines a bundle over B with fibre $F_1 \times F_2$. For a rank-n vector bundle V over B there is a map $V \times_B FV \to B \times \mathbb{R}^n$ given by evaluating a frame on a vector, which extends to the fibre-wise compactifications to give a map

$$eval\colon (V \times_B FV)^+ \to \underline{S^n},$$

where $\underline{S^n}$ denotes the trivial bundle $B \times S^n$ over B. Let V, W be rank-n vector bundles with Riemannian metrics over the space B. A bundle map $\xi\colon V \to W$ which preserves the metric on each fibre gives rise to a map $\xi^+\colon (V \times_B FV)^+ \to (W \times_B FW)^+$ in the obvious way and the diagram

$$\begin{tikzcd}
(V \times_B FV)^+ \arrow[rr, "\xi^+"] \arrow[dr, "eval"'] & & (W \times_B FW)^+ \arrow[dl, "eval"] \\
& B \times S^n &
\end{tikzcd}$$

is commutative. Let $\|\cdot\|_x$ denote the norm induced on the fibre V_x of V by the Riemannian metric. Then $\|\cdot\|_x$ induces a map $(V \times_B FV)_x^+ \to [0,\infty]$, also denoted by $\|\cdot\|_x$ and referred to as *norm* on the fibre $(V \times_B FV)_x^+$, and $\|\cdot\|_\bullet$ is continuous as a map $(V \times_B FV)^+ \to [0,\infty]$. The map ξ^+ preserves this norm, i. e. $\|\xi^+(v)\|_x = \|v\|_x$ for all $v \in (V \times_B FV)\big|_x^+$.

Lemma 6.1 *Let V, W be vector bundles over B with Riemannian metrics, let $U \subset B$ be open, $B_0 = B \setminus U$, and let $\xi \colon V|_U \to W|_U$ be a bundle isomorphism which preserves the metric on each fibre. Then ξ induces an isomorphism*

$$\bar{\xi} \colon \Gamma\left((B, B_0), ((V \times_B FV)^+, \infty)\right) \to \Gamma\left((B, B_0), ((W \times_B FW)^+, \infty)\right).$$

Proof: To simplify notations, we identify B with the ∞-sections of $(V \times_B FV)^+$ and $(W \times_B FW)^+$. We construct a map

$$\xi^+ \colon (V \times_B FV)\Big|_U^+ \cup B \longrightarrow (W \times_B FW)\Big|_U^+ \cup B$$

by defining it in the same way as before on $(V \times_B FV)\Big|_U^+$ and as the identity on B. This map is clearly well-defined. We show that it is continuous. This is clear except possibly for points in ∂U. Let $p \in \partial U$. Choose a neighbourhood $\mathcal{U}$ of p in B such that $V\Big|_{\mathcal{U}}$ and $W\Big|_{\mathcal{U}}$ are trivial. Let $\mathcal{N}$ be a system of neighbourhoods of p in B. For $N \in \mathcal{N}$ and $\varepsilon > 0$, let

$$\mathcal{V}_{N,\varepsilon} = \left\{ x \in (V \times_B FV)\Big|_{N \cap \mathcal{U}}^+ \;\Big|\; \|x\| > \frac{1}{\varepsilon} \right\}.$$

The sets $\mathcal{V}_{N,\varepsilon}$ for $\varepsilon > 0$, $N \in \mathcal{N}$ form a system of neighbourhoods of $p \in (V \times_B FV)^+$. Let $\mathcal{V}^\circ_{N,\varepsilon} = \mathcal{V}_{N,\varepsilon} \cap \left((V \times_B FV)\Big|_U^+ \cup B\right)$ and $\mathcal{W}^\circ_{N,\varepsilon}$ be defined in the analogous way, replacing V by W. One verifies that, for $N \in \mathcal{N}$ and $\varepsilon > 0$, ξ^+ maps $\mathcal{V}^\circ_{N,\varepsilon}$ to $\mathcal{W}^\circ_{N,\varepsilon}$, so ξ^+ is continuous at p. An inverse map is constructed in the same way. Now we define the map $\bar{\xi}$ as composing a section with ξ^+. □

Lemma 6.2 *Let V be a vector bundle with a Riemannian metric over the compact base space B, let $U \in B$ be open and let $K = B \setminus U$. Then the inclusion*

$$i \colon \Gamma^c\left(U, (V \times_B FV)\Big|_U^+\right) \longrightarrow \Gamma\left((B, K), ((V \times_B FV)^+, \infty)\right)$$

is a homotopy equivalence.

Proof: We construct a homotopy inverse j of i as follows. Define a map $h: [0,\infty] \times [0,1] \to [0,\infty]$ by the formula

$$h(x,t) = \begin{cases} \infty & \text{for } x \geq \frac{1}{t} \\ \frac{x}{1-tx} & \text{otherwise} \end{cases}$$

One checks that h is continuous. Now define a map

$$H: (V \times_B FV)^+ \times [0,1] \to (V \times_B FV)^+$$

by the formula $H((v,f),t) = (h(\|v\|,t) \cdot v, f)$. Write H_t for $H(\,\cdot\,,t)$. Notice that H_0 is the identity and that $H_1(v) = \infty$ for $\|v\| > 1$. Define the map

$$j: \Gamma\Big((B,K),((V \times_B FV)^+,\infty)\Big) \longrightarrow \Gamma^c\Big(U,(V \times_B FV)\Big|_U^+\Big)$$

by the formula $j(\sigma) = (H_1 \circ \sigma)\Big|_U$. Let $W = \Big\{x \in B \,\Big|\, \|\sigma(x)\| > 1\Big\}$. Then W is open, $K \subset W$, and $\|H_1(\sigma(x))\| = \infty$ for all $x \in W$, so the support of $H_1 \circ \sigma$ is contained in $B \setminus W$, which is a compact subset of U. This shows that $j(\sigma)$ has compact support.

The composition $i \circ j$ is just the map given by $\sigma \mapsto H_1 \circ \sigma$. This map is homotopic to $\sigma \mapsto H_0 \circ \sigma$, which is the identity. The reverse composition $j \circ i$ is given by composing a section with the appropriate restriction of H_1, which is just in the same way seen to be homotopic to the identity, so j is a homotopy inverse of i. $\square$

Let M be a closed, orientable n-dimensional manifold and let $F \to E \to M$ be a fibre bundle where the fibre F is $(n-1)$-connected and $\pi_n(F) \cong \mathbf{Z}$. Suppose that there is a 'zero section' $\sigma^0: M \to E$, i. e. a section such that $\sigma^0(m) = \infty_m$, the basepoint of the fibre over m for all $m \in M$. In this case it is possible to define the *degree* $\deg(\sigma) \in \mathbf{Z}$ for each section σ of the bundle, and the space $\Gamma(M,E)$ has countably many path components, labelled by the degree. This is done in the following way, which directly generalizes the concept of the degree of a map between spaces. From the Leray-Serre spectral sequence of the fibration we get a short-exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(M;\mathcal{H}_n(F)) & \longrightarrow & H_n(E) & \longrightarrow & H_n(M;\mathcal{H}_0(F)) & \longrightarrow & 0 \\ & & \wr\Vert & & & & \wr\Vert & & \\ & & H_n(F) & & & & H_n(M) & & \end{array}$$

The map $\sigma^0_*\colon H_n(M) \to H_n(E)$ defines a splitting of this sequence and a unique map $d\colon H_n(E) \to H_n(F)$. Choose generators $\phi \in H_n(F)$ and $\mu \in H_n(M)$. Now let σ be any section $M \to E$. Define the degree of σ by the formula

$$d \circ \sigma_*(\mu) = \deg(\sigma) \cdot \phi. \tag{6.1}$$

This definition, of course, depends on the choice of the zero section σ^0, which corresponds to the choice of a basepoint in the pointed set $\pi_0(\Gamma(M, E))$, and clearly $\deg(\sigma^0) = 0$. It is also clear that two homotopic sections have the same degree. The converse can be proved using obstruction theory. We will write $\Gamma_k(M, E)$ for the space of sections of degree k.

In particular, the above applies to the situation where V is a rank-n vector bundle with Riemannian metric over M and $E = (V \times_M FV)^+$. Let $*$ be the basepoint of M. Then, according to lemma 6.2, the inclusion $i\colon \Gamma^c(M \setminus *, E) \to \Gamma^\bullet(M, E)$ is a homotopy equivalence, so both spaces have the same path components. For $\sigma \in \Gamma^c(M \setminus *, E)$ we define $\deg(\sigma)$ to be $\deg(i(\sigma))$ and write $\Gamma^c_k(M \setminus *, E)$ for the space of sections of degree k with compact support. Similarly, the concept of the degree of a section can be extended to compact manifolds with boundary and the degree of a compactly supported section to open paracompact manifolds.

6.2 Labelled configuration spaces

In [22], Dusa McDuff defines certain configuration spaces associated to a smooth manifold M, and shows how they approximate homologically the spaces of sections of certain bundles associated to M. The most basic example is the space $C(M)$ of all finite subsets of M. It can be thought of as the configuration space of indistinguishable particles on M and is given a topology such that particles cannot collide. As a next step, one can associate to each particle the 'sign' $+$ or $-$, and let particles of opposite signs cancel when they collide. The space $C^{\pm}(M)$ obtained in this way is a simple example of what we call a 'labelled configuration space', a space where particles have some internal structure, expressed by a parameter with values in a parameter space P. In the situation we are concerned with, this parameter space varies from point to point of M, forming a fibre bundle over M. The topology of all these configuration spaces is closely related to the spaces of sections of certain fibre bundles over M. This follows in principle from the methods in [22], but details are given there only for the spaces $C(M)$ and $C^{\pm}(M)$. In this section we provide some details for the case relevant to us.

Let M be an n-dimensional Riemannian manifold with basepoint $*$. For $k \in \mathbb{N}$, define the ***space of configurations*** of k points in M as

$$C_k(M) = \left\{ A \subset M \;\middle|\; \operatorname{card}(A) = k \right\},$$

topologized as a subquotient of M^k, and let

$$C(M) = \coprod_{k \in \mathbb{N}} C_k(M).$$

Let $F \hookrightarrow E \xrightarrow{p} M$ be a smooth fibre bundle. We define the space of configurations of k points in M ***with labels in*** E as

$$C_k(M; E) = \left\{ A \in C_k(E) \;\middle|\; p(A) \in C_k(M) \right\},$$

and again let

$$C(M;E) = \coprod_{k \in \mathbb{N}} C_k(M;E).$$

Notice that, if $SP^k(M)$ denotes the k-fold symmetric product of M, the space $C_k(M;E)$ is determined by the pull-back diagram of fibrations

$$\begin{tikzcd}
& F^k \arrow[dl, hook] \arrow[dr, hook] & \\
C_k(M;E) \arrow[rr] \arrow[d] & & SP^k(E) \arrow[d] \\
C_k(M) \arrow[rr] & & SP^k(M)
\end{tikzcd}$$

The spaces $C_k(M)$ and $C_k(M;E)$ are smooth manifolds and the bundle $F^k \hookrightarrow C_k(M;E) \to C_k(M)$ is a smooth fibre bundle. If $M_0 \subset M$ then we sometimes write $C_k(M_0;E)$ instead of $C_k(M_0;E|_{M_0})$, if no ambiguity arises from this notation.

Let TM be the tangent bundle of M. As in section 6.1, we form its fibre-wise one-point compactification T^+M. In [22], the following theorems are proved.

Theorem 6.3 ([22, thm. 1.1]) *Let M be closed compact manifold. Then there are maps*

$$C_k(M) \to \Gamma_k(M, T^+M)$$

and for each $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for $k \geq q$ the induced map

$$H_n(C_k(M)) \to H_n(\Gamma_k(M, T^+M))$$

is an isomorphism.

Theorem 6.4 ([22, thm. 1.2]) *Let M be an open paracompact manifold (i. e. it has no closed components). Then there are maps*

$$C_k(M) \to C_{k+1}(M),$$

$$\Gamma^c_k(M, T^+M) \to \Gamma^c_{k+1}(M, T^+M)$$

and

$$C_k(M) \to \Gamma^c_k(M, T^+M)$$

which induce an isomorphism

$$\varinjlim_{k\to\infty} H_*(C_k(M)) \cong \varinjlim_{k\to\infty} H_*(\Gamma^c_k(M, T^+M)).$$

Moreover, if M is the interior of a compact manifold with boundary, then for each $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for $k \geq q$ the map

$$H_n(C_k(M)) \to H_n(\Gamma^c_k(M, T^+M))$$

is an isomorphism.

Let M be orientable and let FM denote the bundle of orthonormal oriented frames of TM. We are concerned with spaces of configurations of points in M with labels in FM. The corresponding theorems in our case are the following.

Theorem 6.5 *Let M be closed compact. Then there are maps*

$$C_k(M; FM) \to \Gamma_k(M, (TM \times_M FM)^+)$$

and for each $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for $k \geq q$ the induced map

$$H_n(C_k(M; FM)) \to H_n(\Gamma_k(M, (TM \times_M FM)^+))$$

is an isomorphism.

Theorem 6.6 *Let M be open, paracompact. Then there are maps*

$$C_k(M; FM) \to C_{k+1}(M; FM),$$

$$\Gamma_k^c(M, (TM \times_M FM)^+) \to \Gamma_{k+1}^c(M, (TM \times_M FM)^+)$$

and

$$C_k(M; FM) \to \Gamma_k^c(M, (TM \times_M FM)^+)$$

which induce an isomorphism

$$\varinjlim_{k\to\infty} H_*(C_k(M; FM)) \cong \varinjlim_{k\to\infty} H_*(\Gamma_k^c(M, (TM \times_M FM)^+)).$$

Moreover, if M is the interior of a compact manifold with boundary, then for each $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for $k \geq q$ the map

$$H_n(C_k(M; FM)) \to H_n(\Gamma_k^c(M, (TM \times_M FM)^+))$$

is an isomorphism.

In fact, the only theorem we will appeal to later is theorem 6.6. It is proved by essentially the same methods as theorem 6.4. For the sake of completeness we give a sketch of the proof, pointing out the necessary alterations as we go along.

Let $L \subset M$ be a closed subset. Define an equivalence relation on $C(M)$ by $A \sim B$ if $A \cap (M \setminus L) = B \cap (M \setminus L)$ for $A, B \subset M$. Let $C(M, L)$ be the quotient $C(M)/\sim$. Similarly, for a smooth fibre bundle $F \hookrightarrow E \xrightarrow{p} M$, define $C(M, L; E)$ as $C(M; E)/\approx$ where the equivalence relation is given by $A \approx B$ if $A \cap E\big|_{M\setminus L} = B \cap E\big|_{M\setminus L}$ for $A, B \in C(M; E)$. We write $\bar{C}(M)$ for $C(M, \partial M)$ and $\bar{C}(M; E)$ for $C(M, \partial M; E)$. One checks that all the configuration spaces defined in this way have the homotopy type of CW complexes.

The first step in the proof of theorems 6.4 and 6.6 is to construct an appropriate map $\tilde{C}(M) \to \Gamma(M, T^+M)$ and $\tilde{C}(M; FM) \to \Gamma(M, (TM \times_M FM)^+)$. Let d denote the Riemannian metric on M. For $\varepsilon > 0$ let $M_\varepsilon = \{x \in M \mid d(x, \partial M) \geq \varepsilon\}$. Choose ε sufficiently small for $M \setminus M_{2\varepsilon}$ to be homeomorphic to $\partial M \times [0, 2\varepsilon[$. Let

$$\tilde{C}_\varepsilon(M) = \left\{([A], \delta) \in \tilde{C}(M) \times]0; \varepsilon] \,\middle|\, d(x,y) \geq 2\delta \text{ for all } x, y \in A, x \neq y\right\}.$$

The projection map $\tilde{C}_\varepsilon(M) \to \tilde{C}(M)$ is a homotopy equivalence, and so is the restriction map $\Gamma(M, T^+M) \to \Gamma(M_\varepsilon, T^+M)$. Hence it is sufficient to construct a map $\phi_\varepsilon \colon \tilde{C}_\varepsilon(M) \to \Gamma(M_\varepsilon, T^+M)$. Suppose that ε is small enough so that for each $x \in M_\varepsilon$ the exponential map defines a diffeomorphism between the disk of diameter ε in T_xM and $\overline{B}_\varepsilon(x) = \{y \in M \mid d(x,y) \leq \varepsilon\}$. Let $([A], \delta) \in \tilde{C}_\varepsilon(M)$. For $x \in M_\varepsilon$ there is at most one element $y \in A$ such that $d(x,y) < \delta$. If such $y \in A$ exists, let $t(x,y)$ be the unit tangent at x to the minimal geodesic from x to y. Define

$$\phi_\varepsilon([A], \delta)(x) = \begin{cases} \dfrac{d(x,y)}{\delta - d(x,y)} \cdot t(x,y) & \text{if } y \text{ exists;} \\ \infty_x & \text{otherwise.} \end{cases}$$

Let $p : FM \to M$ be the projection. In an analogous way to the above construction, let $\tilde{C}_\varepsilon(M; E)$ be the space

$$\left\{([A], \delta) \in \tilde{C}(M; E) \times]0; \varepsilon] \,\middle|\, d(p(x), p(y)) \geq 2\delta \text{ for all } x, y \in A, x \neq y\right\}.$$

Again, it is sufficient to construct a map $\phi_\varepsilon \colon \tilde{C}_\varepsilon(M; FM) \to \Gamma(M_\varepsilon, (TM \times_M FM)^+)$. Choose a connection ∇ on FM. Let $([A], \delta) \in \tilde{C}_\varepsilon(M; FM)$. Again, there exists at most one $z \in A$ such that $d(p(z), x) < \delta$. If such z exists, let $y = p(z)$. For $u, v \in M$, $d(u,v) < \varepsilon$, let $\nabla_{u,v} \colon F_uM \to F_vM$ be given by parallel transport with respect to ∇ along the minimal geodesic from u to v. Define

$$\phi_\varepsilon([A], \delta)(x) = \begin{cases} \left(\frac{d(x,y)}{\delta - d(x,y)} \cdot t(x,y),\, \nabla_{y,x}(z)\right) & \text{if } z \text{ exists;} \\ \infty_x & \text{otherwise.} \end{cases}$$

Notice that the definition of the map ϕ_ε depends on the choice of the metric d and in the second case also of the connection ∇. Different choices of d and ∇ give rise to homotopic maps.

Lemma 6.7 (cf. [22, thm. 2.5]) *If M is compact and has no closed components, then the maps*

$$\phi_\varepsilon : \tilde{C}_\varepsilon(M) \to \Gamma(M_\varepsilon, T^+M)$$

and

$$\phi_\varepsilon : \tilde{C}_\varepsilon(M; FM) \to \Gamma(M_\varepsilon, (TM \times_M FM)^+)$$

defined above are homotopy equivalences.

Proof: Suppose first that $M = D$, the closed unit disk in $\mathbf{R}^n$. For $\lambda > 0$, let

$$\tilde{C}(D, \lambda) = \left\{ [A] \in \tilde{C}(D) \,\middle|\, d(x, y) \geq 2\lambda \text{ for } x, y \in A,\ x \neq y \right\}$$

and

$$\tilde{C}(D, \lambda; FD) = \left\{ [A] \in \tilde{C}(D; FD) \,\middle|\, d(p(x), p(y)) \geq 2\lambda \text{ for } x, y \in A,\ x \neq y \right\}.$$

Let $\delta \leq \varepsilon$. The maps $\tilde{C}(D, \delta) \to \tilde{C}(D, 2)$ and $\tilde{C}(D, \delta; FD) \to \tilde{C}(D, 2; FD)$ (in the following diagram denoted by ξ) induced by the radial expansion $x \mapsto \frac{2}{\delta}x$ of the unit disk are homotopy equivalences. Notice that the second of these maps depends again on the choice of the connection ∇. Since configurations in $\tilde{C}(D, 2)$ consist of at most one particle, there are canonical isomorphisms $\tilde{C}(D, 2) \cong S^n$ and $\tilde{C}(D, 2; FD) \cong (\mathbf{R}^n \times SO(n))^+$. On the other hand, evaluation at the origin of D gives homotopy equivalences $\Gamma(D_\varepsilon, T^+D_\varepsilon) \xrightarrow{\simeq} S^n$ and $\Gamma(D_\varepsilon, (TD_\varepsilon \times_{D_\varepsilon} FD_\varepsilon)^+) \xrightarrow{\simeq} (\mathbf{R}^n \times SO(n))^+$, and the following two diagrams are homotopy commutative, where the top horizontal maps are induced by ϕ_ε.

$$\begin{array}{ccc}
\tilde{C}(D,\delta) & \longrightarrow & \Gamma(D_\varepsilon, T^+D_\varepsilon) \\
\Big\downarrow \xi & & \Big\downarrow \simeq \\
\tilde{C}(D,2) & \xrightarrow[\cong]{} & S^n
\end{array}
\qquad
\begin{array}{ccc}
\tilde{C}(D,\delta;FD) & \longrightarrow & \Gamma(D_\varepsilon, (TD_\varepsilon \times_{D_\varepsilon} FD_\varepsilon)^+) \\
\Big\downarrow \xi & & \Big\downarrow \simeq \\
\tilde{C}(D,2;FD) & \xrightarrow[\cong]{} & (\mathbf{R}^n \times SO(n))^+
\end{array}$$

One concludes that the top horizontal maps are homotopy equivalences. It follows that the maps ϕ_ε are also equivalences. This proves the lemma for the case $M = D$.

Next one can show that for certain inclusions $N \subset M$ of a manifold N of the same dimension as M, the restriction maps $\tilde{C}(M) \to \tilde{C}(N)$ and $\tilde{C}(M;FM) \to \tilde{C}(N;FN)$ are *quasifibrations*, i. e. the inclusion of the actual fibres into the homotopy fibres induces an isomorphism on homotopy groups. Using this fact, one can, for certain submanifolds M_1 and M_2 of M, prove that $\tilde{C}(M_1 \cup M_2)$ and $\tilde{C}(M_1 \cup M_2; F(M_1 \cup M_2))$ have the homotopy type of the fibre products

$$\tilde{C}(M_1) \times_{\tilde{C}(M_1 \cap M_2)} \tilde{C}(M_2)$$

and

$$\tilde{C}(M_1; FM_1) \times_{\tilde{C}(M_1 \cap M_2; F(M_1 \cap M_2))} \tilde{C}(M_2; FM_2)$$

respectively.

This allows us to prove lemma 6.7 by building M from simpler pieces. First one shows that it holds for $S^k \times D^{n-k}$ $(k < n)$ by induction on k, starting from $S^0 \times D^n$, for which we have already proved the lemma. (For $k = n$ the above methods fail, because some inclusions occur that do not have the right properties. One can even see that the statement would be false in this case.) Then one proves the lemma by induction on the number of handles in a handlebody decomposition of M. Again, one has to require the index of the handles to be less than n, which causes no problem, if M is compact with no closed components. □

Lemma 6.8 (cf. [22, thm. 2.6]) *Let M be compact, connected, and suppose that ∂M is the union of two submanifolds $L \neq \emptyset$ and L', of the same dimension as ∂M and with boundaries $\partial L = \partial L' = L \cap L'$. Then there are homotopy equivalences*

$$C(M, L) \to \Gamma\left((M, L'), (T^+M, \infty)\right)$$

and

$$C(M, L; FM) \to \Gamma\left((M, L'), ((TM \times_M FM)^+, \infty)\right).$$

Notice that taking $L = \partial M$ and $L' = \emptyset$, we recover the statement of lemma 6.7.

Proof: First suppose that L' has no closed components. Let $I = [0, 1]$ and identify $L' \subset \partial M$ with $L' \times \{0\}$. Let $X = M \cup_{L'} (L' \times I)$. Then there are homotopy commutative diagrams

$$\begin{array}{ccc} \bar{C}(X) & \longrightarrow & \Gamma(X, T^+X) \\ \downarrow r & & \downarrow \\ \bar{C}(L' \times I) & \longrightarrow & \Gamma(L' \times I, T^+X) \end{array} \qquad \begin{array}{ccc} \bar{C}(X; FX) & \longrightarrow & \Gamma(X, (TX \times_X FX)^+) \\ \downarrow r & & \downarrow \\ \bar{C}(L' \times I; FX) & \longrightarrow & \Gamma(L' \times I, (TX \times_X FX)^+) \end{array}$$

where the maps r can again be shown to be quasifibrations. Notice that X is homeomorphic to M and the horizontal maps are equivalences according to lemma 6.7. The resulting equivalences between the fibres of the vertical maps give the result.

In the general case, let $L' = A' \cup B'$, where A' is the union of all the closed components of L'. Let $A = \partial M \setminus A'$. First one proves that

$$C(M, A) \simeq \Gamma((M, A'), (T^+M, \infty))$$

and

$$C(M, A; FM) \simeq \Gamma((M, A'), ((TM \times_M FM)^+, \infty)).$$

This is done in the same way as in lemma 6.7. The theorem is clearly true for $M = A' \times I$, where A' is identified with $A' \times \{0\}$, because the spaces in question are contractible. Then one builds M by attaching a finite number of handles to $A' \times I$. Let $X = M \cup_{B'} (B' \times I)$. Now $(X, \partial X \setminus A') \cong (M, A)$, and the result is obtained by a similar argument as above, replacing $\bar{C}(X)$ by $C(X, \partial X \setminus A')$, $\Gamma(X, T^+X)$ by $\Gamma((X, A'), (T^+X, \infty))$, L' by B' etc. □

As in the above proof, the treatment of labelled and unlabelled configuration spaces is completely the same in the following arguments. We will only give the labelled version. Let M be a compact connected manifold with non-empty boundary ∂M. Choose $m \in \partial M$ and $\overline{m} \in F_m M$. Construct a map

$$g\colon C_k(M; FM) \to C_{k+1}(M; FM)$$

in the following way. Let $f_t\colon M \to M$ be an injective homotopy such that $f_0 = \mathrm{id}$, $f_t = \mathrm{id}$ except near m, and $m \notin f_1(M)$. By parallel transport along the trajectories of f, using the connection ∇, one obtains a homotopy $F_t\colon FM \to FM$. For $A \in M$ then define $g(A) = F_1(A) \cup \{\overline{m}\}$.

There are maps $C_k(M; FM) \to C(M, \{m\}; FM)$, and one can show that they induce an isomorphism

$$\varinjlim_{k\to\infty} H_*(C_k(M; FM)) \cong H_*(C(M, \{m\}; FM)).$$

The space $C(M, \{m\}; FM)$ can be related to a space of sections as follows. Let L be the component of ∂M containing m and $L' = \partial M \setminus L$. Let $A = L \setminus \{m\} \times I$. Then ∂A consists of the pieces $A_0 = (L \setminus \{m\}) \times \{0\}$ and $A_1 = (L \setminus \{m\}) \times \{1\}$. Let $X = M \cup_{A_0} A$. The fibres of the restriction map

$$C(X, A_1 \cup \{m\}; FX) \to \bar{C}(A; FA)$$

are homeomorphic to $C(M,\{m\};FM)$ and one can show that this map is a *homology fibration*, i. e. the inclusion of its actual fibres into its homotopy fibres induces an isomorphism in homology. Using lemma 6.8, one shows that

$$\tilde{C}(A;FA) \simeq \Gamma((L,\{m\}),((TM \times_M FM)^+,\infty)).$$

Since the space $C(X,A_1 \cup \{m\};FX)$ does not have a simple description in terms of spaces of sections, one passes to certain covering spaces $\overline{C}(X,A_1 \cup \{m\};FX)$ and $\overline{C}(A;FA)$. The covering $\overline{C}(X,A_1 \cup \{m\};FX)$ turns out to be homotopy equivalent to $C(M\setminus\{m\},L\setminus\{m\};FM)$, which, using again lemma 6.8, can be seen to be homotopy equivalent to

$$\Gamma((M,\{m\} \cup L'),((TM \times_M FM)^+,\infty).$$

The covering $\overline{C}(A;FA)$ turns out to be homotopy equivalent to the universal cover of $\tilde{C}(A;FA)$, and the induced map $\overline{C}(X,A_1 \cup \{m\};FX) \to \overline{C}(A;FA)$ is again a homology fibration with fibre $C(M,\{m\};FM)$. Putting all these ingredients together, one arrives at the next lemma.

Lemma 6.9 (cf. [22, thm. 4.5]) *Let M be a connected compact manifold with non-empty boundary ∂M. Then there is a map*

$$C(M;FM) \to \Gamma((M,\partial M),((TM \times_M FM)^+,\infty))$$

which takes $C_k(M;FM)$ into $\Gamma_k((M,\partial M),((TM \times_M FM)^+,\infty))$ for each k, and induces an isomorphism

$$\varinjlim_k H_*(C_k(M;FM)) \cong \varinjlim_k H_*\left(\Gamma_k((M,\partial M),((TM \times_M FM)^+,\infty))\right).$$

Moreover, for each $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that

$$H_n(C_k(M;FM)) \to H_n\left(\Gamma_k((M,\partial M),((TM \times_M FM)^+,\infty))\right)$$

is an isomorphism for $k > q$.

Proof: Let $\overline{\Gamma}((L,\{m\}),((TM\times_M FM)^+,\infty))$ denote the universal cover. Then there is a homotopy commutative diagram

$$\begin{array}{ccc}
\overline{C}(X,A_1\cup\{m\};FX) & \xrightarrow{\simeq} & \Gamma((M,\{m\}\cup L'),((TM\times_M FM)^+)) \\
\downarrow & & \downarrow \\
\overline{C}(A;FA) & \xrightarrow{\simeq} & \overline{\Gamma}((L,\{m\}),((TM\times_M FM)^+,\infty))
\end{array}$$

where the two vertical maps are restrictions and the horizontal maps the equivalences discussed above. Since the fibre of the right-hand vertical map is easily seen to be just one connected component of $\Gamma((M,\partial M),((TM\times_M FM)^+,\infty))$ (all of which have the same homotopy type), and the fibre of the left-hand vertical map is $C(M,\{m\};FM)$, we obtain isomorphisms

$$\begin{aligned}
\varinjlim_{k\to\infty} H_*(C_k(M,FM)) &\xrightarrow{\cong} H_*(C(M,\{m\},FM)) \\
&\xrightarrow{\cong} H_*(\Gamma_0((M,\partial M),((TM\times_M FM)^+))).
\end{aligned}$$

Next, one shows that there are maps

$$\Gamma_k((M,\partial M),((TM\times_M FM)^+))\to\Gamma_{k+1}((M,\partial M),((TM\times_M FM)^+)),$$

obtained by adding a section of degree 1 near m. The inclusion of the degree-zero component into the direct limit is a homotopy equivalence. Also the direct system of these section spaces is compatible with the direct system $(C_k(M;FM),g)$, and the induced map

$$\begin{aligned}
\varinjlim_{k\to\infty} H_*(C_k(M;FM)) &\longrightarrow \varinjlim_{k\to\infty} H_*(\Gamma_k((M,\partial M),((TM\times_M FM)^+))) \\
&\cong H_*(\Gamma_0((M,\partial M),((TM\times_M FM)^+)))
\end{aligned}$$

is the same as the one obtained previously, hence it is an isomorphism.

To finish the proof of lemma 6.9 we have to show that the direct system of the groups $H_*(C_k(M;FM))$ stabilizes in any given degree after a finite number of steps. First one shows by a method of A. Dold [8], constructing apropriate 'transfer homomorphisms', that the map $g_*\colon H_*(C_k(M;FM)) \to H_*(C_{k+1}(M;FM))$ is the embedding of a direct summand. The limit, on the other hand, is equal to the homology of the space of sections, which is finitely generated in each degree for compact M, hence the result. □

Proof of theorem 6.6: If M is the interior of a compact manifold with boundary $\overline{M}$, then the inclusion $M \hookrightarrow \overline{M}$ induces a homotopy equivalence $C(M;FM) \hookrightarrow C(\overline{M};F\overline{M})$. Also, according to lemma 6.2, the inclusion

$$\Gamma^c\big(M,(TM \times_M FM)^+\big) \longrightarrow \Gamma\Big(\big(\overline{M},\partial\overline{M}\big),\,\big((T\overline{M} \times_{\overline{M}} F\overline{M})^+,\infty\big)\Big)$$

is a homotopy equivalence. The theorem is then an immediate consequence of lemma 6.9.

In general, filter M by compact connected manifolds M_n. One can choose the maps

$$g_n\colon C_k(M_n;FM) \to C_{k+1}(M_n;FM)$$

in such a way that they homotopy commute with the maps induced by the inclusions $M_n \hookrightarrow M_{n+1}$, so one obtains a map

$$g\colon C_k(M;FM) \to C_{k+1}(M;FM).$$

In a similar way one defines a map between the components of the space of sections, and lemma 6.9, applied for each n, gives the result for the limit. Notice, however, that in this case the homology of the limit need not be finitely generated in any given degree, so we cannot guarantee that the homology in any degree is approximated completely after a finite number of steps. □

6.3 The Results of Atiyah and Jones

Let $P \to S^4$ be a pricipal $SU(2)$ bundle of second Chern class k and, as in section 4.1, let $\tilde{\mathcal{B}}_k$ denote the space of connections modulo based gauge equivalence. Further let $\mathcal{M}_k \subseteq \bar{\mathcal{B}}_k$ be the moduli space of self-dual connections. The inclusion map $\mathcal{M}_k \hookrightarrow \bar{\mathcal{B}}_k$ in homology was first studied by M. Atiyah and J. D. S. Jones. This is their result.

Theorem 6.10 (Atiyah–Jones) *The inclusion $\mathcal{M}_k \hookrightarrow \bar{\mathcal{B}}_k$ induces a map in homology $H_q(\mathcal{M}_k) \to H_q(\bar{\mathcal{B}}_k)$ which, for $k \gg q$, is a projection onto a direct summand.*

Their proof (see [3]) uses the t'Hooft construction of instantons to obtain a map

$$\theta_k : C_k(\mathbf{R}^4) \longrightarrow \mathcal{M}_k.$$

Now notice that theorem 6.4, which for this special case is due to G. Segal, asserts that there is a map $\rho : C_k(\mathbf{R}^4) \to \Omega^4_k S^4$ which is an asymptotic homology equivalence. Further recall from section 4.1 that there is a weak homotopy equivalence

$$\bar{\mathcal{B}}_k \xrightarrow{\simeq} \mathrm{Maps}_k(S^4, BSU(2)) \simeq \Omega^3_k S^3.$$

Atiyah and Jones show that there is a homotopy commutative diagram

$$\begin{tikzcd}
C_k(\mathbf{R}^4) \arrow[r, "\rho"] \arrow[d, "\theta_k"] & \Omega^4_k S^4 \arrow[d] \arrow[dr, "p"] & \\
\mathcal{M}_k \arrow[r, hook] & \bar{\mathcal{B}}_k \arrow[r, "\simeq"] & \Omega^3_k S^3
\end{tikzcd}$$

where p is the map induced by the inclusion $S^4 \hookrightarrow BSU(2)$. Since the map p has a right-inverse given by the suspension map $\Omega^3_k S^3 \to \Omega^4_k S^4$, the result follows. This theorem led to the following conjectures.

i) The map $\mathcal{M}_k \to \bar{\mathcal{B}}_k$ is asymptotically an isomorphism in homotopy and homology.

ii) The same holds for a more general four-manifold.

The first point was settled affirmatively by C. Boyer *et al.* (see [4]), using methods from algebraic geometry. They also gave an explicit function $k(q)$ such that equivalence holds in the range up to q whenever $k \geq k(q)$. Concerning the second point, C. H. Taubes proved that for any closed oriented smooth four-manifold and any connected simple Lie group G, the induced map in homology and homotopy by the inclusion $\mathcal{M}_k \hookrightarrow \tilde{\mathcal{B}}_k$ is asymptotically surjective (see [28]). Taubes' theorem, however, does not make any explicit statement about the range where equivalence holds.

Now let M be a smooth, closed, simply-connected four-manifold with basepoint $*$ and let $M_0 = M \setminus *$. Motivated by the original programme of Atiyah and Jones, our aim is to construct an approximation of the space $\mathrm{Maps}(M, BSU(2))$ by configuration spaces which gives an asymptotic surjection in homology. With theorem 6.6 in mind, our candidate for a configuration space is $C_k(M_0; FM)$, which turns out to be an appropriate model for $\mathrm{Maps}(M, S^4)$. But then the analogy with the case $M = S^4$ fails, because, in general, the map $\mathrm{Maps}(M, S^4) \to \mathrm{Maps}(M, BSU(2))$ does not induce a surjection in homology. As a consequence of the results in chapter 5 we do, however, obtain a surjection in mod 2 homology, as described in the next section.

6.4 The Z/2Z-case

Let M be a smooth, closed, simply-connected Riemannian 4-manifold with basepoint $*$ and let $M_0 = M \setminus *$. Let TM denote the tangent bundle of M, FM its orthonormal frame bundle and T^+M the fibre-wise one-point compactification of TM. Our aim is to construct a map

$$\rho \colon C_k(M_0; FM) \longrightarrow \mathrm{Maps}_k(M, BSU(2)).$$

In section 6.2 we constructed a map

$$\phi\colon C_k(M_0; FM) \longrightarrow \Gamma_k^c(M_0, (TM \times_M FM)^+)$$

and showed that it is asymptotically an isomorphism in homology. The inclusion

$$i\colon \Gamma_k^c(M_0, (TM \times_M FM)^+) \hookrightarrow \Gamma_k^\bullet(M, (TM \times_M FM)^+)$$

is a homotopy equivalence according to lemma 6.2. Evaluating a frame on a tangent vector gives a map $(TM \times_M FM)^+ \longrightarrow \underline{S^4}$ and this induces a map

$$eval\colon \Gamma_k^\bullet(M, (TM \times_M FM)^+) \longrightarrow \Gamma_k^\bullet(M, \underline{S^4}) \cong \mathrm{Maps}_k(M, S^4).$$

Finally, there is a map

$$\pi\colon\ \mathrm{Maps}_k(M; S^4) \longrightarrow \mathrm{Maps}_k(M, BSU(2))$$

induced by the standard inclusion of S^4 in $BSU(2)$. Define the map ρ to be the composition $\pi \circ eval \circ i \circ \phi$.

Remark: This composite can be described as gluing at each of k points in M (none of which is the basepoint) a standard degree 1 map $S^4 \to BSU(2)$ into the zero map $M \to BSU(2)$, using the label (i. e. the framing) at each point to identify a neighbourhood of that point in M with the standard 4-disk.

Theorem 6.11 *Let M be a smooth simply-connected closed four-manifold. Then the induced map in $\mathbb{Z}/2\mathbb{Z}$-homology*

$$\rho_*\colon H_*(C_k(M_0; FM); \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_*(\mathrm{Maps}_k(M, BSU(2)); \mathbb{Z}/2\mathbb{Z})$$

is asymptotically surjective, i. e. for each $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for $k > q$ the map

$$\rho_*\colon H_n(C_k(M_0; FM); \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_n(\mathrm{Maps}_k(M, BSU(2)); \mathbb{Z}/2\mathbb{Z})$$

is surjective.

Proof (for M spin): The map ρ is defined as the composite $\pi \circ eval \circ i \circ \phi$ where ϕ is an asymptotic isomorphism in homology, i is a homotopy equivalence and π is surjective in $\mathbb{Z}/2\mathbb{Z}$-homology according to corollary 5.18. It is therefore sufficient to show that the map $eval$ is surjective in homology. Since M is a spin manifold, the restriction of the tangent bundle to M_0 is trivial. Let $\sigma\colon M_0 \times \mathbf{R}^4 \longrightarrow TM\big|_{M_0}$ be a trivialization respecting the Riemannian metric. This induces a trivialization

$$\sigma^+\colon M_0 \times (\mathbf{R}^4 \times SO(4))^+ \longrightarrow (TM \times_M FM)\big|^+_{M_0}.$$

Let $s\colon \mathrm{Maps}_k(M, S^4) \to \mathrm{Maps}_k(M, (\mathbf{R}^4 \times SO(4))^+)$ be induced by the standard inclusion of $S^4 \cong (\mathbf{R}^4)^+$ into $((\mathbf{R}^4 \times SO(4))^+$. Let

$$\tilde{\sigma}\colon \mathrm{Maps}_k(M, (\mathbf{R}^4 \times SO(4))^+) \longrightarrow \Gamma^\bullet_k(M, (TM \times_M FM)^+)$$

be obtained from σ^+ according to lemma 6.1. The composition $\tilde{\sigma} \circ s$ is a right-inverse to the map $eval$. □

In order to prove theorem 6.11 for manifolds with odd intersection form, we need the following lemma. Let M be a manifold with odd intersection form and let the complex L be obtained from M as described in lemma 2.7.

Lemma 6.12 *There exists a map*

$$\xi\colon \mathrm{Maps}_k(L, (\mathbf{R}^4 \times SO(4))^+) \longrightarrow \Gamma^\bullet_k(M, (TM \times_M FM)^+)$$

such that the following diagram is homotopy commutative

$$(6.2)\qquad \begin{array}{ccc} \mathrm{Maps}_k(L, (\mathbf{R}^4 \times SO(4))^+) & \xrightarrow{\ \xi\ } & \Gamma^\bullet_k(M, (TM \times_M FM)^+) \\ \Big\downarrow{\scriptstyle eval} & & \Big\downarrow{\scriptstyle eval} \\ \mathrm{Maps}_k(L, S^4) & \xrightarrow{\ \bar{p}\ } & \mathrm{Maps}_k(M, S^4) \end{array}$$

where $\bar{p}$ is induced by the projection $p\colon M \to L$.

Proof: Choose a surface $\Sigma \subset M$ as in corollary 2.12 and recall that the map p factorizes up to homotopy through M/Σ. Hence we obtain a map

$$f\colon \operatorname{Maps}_k(L, (\mathbf{R}^4 \times SO(4))^+) \longrightarrow \operatorname{Maps}_k((M, \Sigma), ((\mathbf{R}^4 \times SO(4))^+, \infty))$$

such that the following diagram is commutative.

$$\begin{tikzcd}
\operatorname{Maps}_k(L, (\mathbf{R}^4 \times SO(4))^+) \arrow[r, "f"] \arrow[d, "eval"] & \operatorname{Maps}_k((M, \Sigma), ((\mathbf{R}^4 \times SO(4))^+, \infty)) \arrow[d, "eval"] \\
\operatorname{Maps}_k(L, S^4) \arrow[r, "op"] & \operatorname{Maps}_k(M, S^4)
\end{tikzcd}$$

Recall from corollary 2.12 that the restriction of TM to the complement of Σ is trivial. Hence we obtain from lemma 6.1 an isomorphism

$$g\colon \operatorname{Maps}_k((M, \Sigma), ((\mathbf{R}^4 \times SO(4))^+, \infty)) \to \Gamma_k((M, \Sigma), (TM \times_M FM)^+, \infty)$$

where the second term maps by the natural inclusion to $\Gamma_k^\bullet(M, (TM \times_M FM)^+)$. Define ξ to be $g \circ f$, followed by this inclusion. Since all the maps involved commute with evaluation, the diagram (6.2) is commutative. □

Proof of theorem 6.11 (for M not spin): Since the map ρ is defined as the composite $\pi \circ eval \circ i \circ \phi$, where ϕ is an asymptotic isomorphism in homology and i is a homotopy equivalence, it is sufficient to show that $\pi \circ eval$ is surjective in $\mathbf{Z}/2\mathbf{Z}$-homology. Consider the following diagram, where the top square is just diagram (6.2) of lemma 6.12.

$$\begin{tikzcd}
\mathrm{Maps}_k(L,(\mathbb{R}^4 \times SO(4))^+) \arrow[r, "\xi"] \arrow[d, "eval"] & \Gamma_k^\bullet(M,(TM \times_M FM)^+) \arrow[d, "eval"] \\
\mathrm{Maps}_k(L,S^4) \arrow[r, "\tilde{p}"] \arrow[d, "\pi"] & \mathrm{Maps}_k(M,S^4) \arrow[d, "\pi"] \\
\mathrm{Maps}_k(L,BSU(2)) \arrow[r, "\tilde{p}"] & \mathrm{Maps}_k(M,BSU(2))
\end{tikzcd}$$

Both left-hand vertical arrows represent maps that are surjective in $\mathbb{Z}/2\mathbb{Z}$-homology. For the first arrow this holds, because the map has an obvious right-inverse. For the second arrow the surjectivity follows from corollary 5.18. As in the proof of theorem 5.17, one shows that the bottom horizontal arrow represents a surjective map in $\mathbb{Z}/2\mathbb{Z}$-homology. Hence the composition of the two right-hand vertical maps is surjective in $\mathbb{Z}/2\mathbb{Z}$-homology as well, and the theorem follows. □

6.5 The Cokernel of the Approximation

In section 6.4 we constructed a map

$$\rho\colon C_k(M_0; FM) \longrightarrow \mathrm{Maps}_k(M, BSU(2))$$

and showed that it is asymptotically surjective in homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. We now show that for other coefficient systems this is not always the case. For convenience we work with cohomology instead of homology. Our aim is to identify an integer class in $H^4(\mathrm{Maps}_k(M, BSU(2)))$ which lies in the kernel of ρ^* for all $k \in \mathbb{N}$. First we need the following definitions. Let $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$. For $j \in \mathbb{Z}$ let

$$ev\colon Y \times \mathrm{Maps}_j(Y, BSU(2)) \longrightarrow BSU(2)$$

be the evaluation map. Let $n \in \mathbb{N}$ and let c be the standard generator of $H^4(BSU(2))$. Recall that $H^*(BSU(2)) \cong \mathbb{Z}[c]$. For $k \in \mathbb{N}$ define the map

$$\mu_n : H_k(Y) \longrightarrow H^{4n-k}\big(\mathrm{Maps}_j(Y, BSU(2))\big)$$

by $a \mapsto ev^*(c^n)/a$ where

$$/ : H^m(Y \times \mathrm{Maps}_j(Y, BSU(2))) \otimes H_k(Y) \longrightarrow H^{m-k}(\mathrm{Maps}_j(Y, BSU(2)))$$

is the slant product. The map μ_1 was introduced by S. K. Donaldson in [10] in order to construct certain cohomology classes in the space of connections modulo gauge equivalence.

Lemma 6.13 *There exists a map*

$$\overline{\mu}_n : H_k(Y) \longrightarrow H^{4n-k}\big(\mathrm{Maps}(X, BSU(2))\big)$$

such that $\mu_n = r^* \circ \overline{\mu}_n$ *where* $r\colon \mathrm{Maps}_j(Y, BSU(2)) \to \mathrm{Maps}(X, BSU(2))$ *is the restriction map.*

Proof: Construct $\overline{\mu}$ in the following way. Let $K(\mathbb{Z}, 4)$ be an Eilenberg-MacLane space and $\gamma \in H^4(K(\mathbb{Z}, 4))$ the standard generator. Define a map

$$\tilde{\mu} : H_k(Y) \longrightarrow H^{4n-k}\big(\mathrm{Maps}_j(Y, K(\mathbb{Z}, 4))\big)$$

by $a \mapsto \widetilde{ev}^*(\gamma^n)/a$ where $\widetilde{ev}$ is the appropriate evaluation map. Notice that the restriction map

$$\tilde{r} : \mathrm{Maps}_j(Y, K(\mathbb{Z}, 4)) \longrightarrow \mathrm{Maps}(X, K(\mathbb{Z}, 4))$$

is a homotopy equivalence. Finally, let $\tilde{\gamma}\colon BSU(2) \to K(\mathbb{Z}, 4)$ be a classifying map for c and $\overline{\gamma}\colon \mathrm{Maps}(X, BSU(2)) \to \mathrm{Maps}(X, K(\mathbb{Z}, 4))$ be given by composition on the left with $\tilde{\gamma}$. Define $\overline{\mu}_n$ as $\overline{\gamma}^* \circ (\tilde{r}^*)^{-1} \circ \tilde{\mu}$. By definition,

$\bar{\gamma}^*(\gamma^n) = c^n$, and from the commutativity of the diagram

$$\begin{tikzcd}
Y \times \mathrm{Maps}_j(Y, BSU(2)) \arrow[r, "ev"] \arrow[d, "\mathrm{id} \times \hat{\gamma}"] & BSU(2) \arrow[d, "\bar{\gamma}"] \\
Y \times \mathrm{Maps}_j(Y, K(\mathbf{Z}, 4)) \arrow[r, "\widetilde{ev}"] & K(\mathbf{Z}, 4)
\end{tikzcd}$$

where $\hat{\gamma}$ is given by composition on the left with $\bar{\gamma}$ we see that $\mu_n = \hat{\gamma}^* \circ \tilde{\mu}_n$. From the commutative diagram

$$\begin{tikzcd}
\mathrm{Maps}_j(Y, BSU(2)) \arrow[r, "r"] \arrow[d, "\hat{\gamma}"] & \mathrm{Maps}(X, BSU(2)) \arrow[d, "\bar{\gamma}"] \\
\mathrm{Maps}_j(Y, K(\mathbf{Z}, 4)) \arrow[r, "\tilde{r}"] & \mathrm{Maps}(X, K(\mathbf{Z}, 4))
\end{tikzcd}$$

we conclude that $\hat{\gamma}^* = r^* \circ (\bar{\gamma})^* \circ (\tilde{r}^*)^{-1}$. □

Remark: Notice that the definition of $\overline{\mu}_n$ depends on the choice of j. One can, however, define similar maps

$$\overline{\overline{\mu}}_n \colon H_k(X) \longrightarrow H^{4n-k}(\mathrm{Maps}(X, BSU(2)))$$

using the evaluation $X \times \mathrm{Maps}(X, BSU(2)) \to BSU(2)$ independent of j. Then $\overline{\overline{\mu}}_n = \overline{\mu}_n \circ \iota_*$, where $\iota\colon X \hookrightarrow Y$ is the inclusion, and it follows that only the restriction of $\overline{\mu}_n$ to $H_4(Y)$ depends on j. In fact, one checks that $\overline{\mu}_1([Y]) = j \in H^0(\mathrm{Maps}(X, BSU(2))) = \mathbf{Z}$ where $[Y]$ is the orientation class. We will see below that $\overline{\mu}_n$ is independent of j for $n \geq 2$.

In lemma 4.3 we established an isomorphism

$$\psi\colon \pi_2(\mathrm{Maps}(X, BSU(2))) \longrightarrow H^2(X)$$

using the evaluation. Comparing this to the map

$$\overline{\overline{\mu}}_1 \colon H_2(X) \longrightarrow H^2(\mathrm{Maps}(X, BSU(2)))$$

we see that it is precisely the dual construction and for $a \in H_2(Y)$, $\beta \in H_2(\mathrm{Maps}(X, BSU(2)))$, there is the relation $\langle \overline{\mu}_1(a), \beta \rangle = \langle \psi(\beta), a \rangle$. Here we identified the second homology and homotopy of $\mathrm{Maps}(X, BSU(2))$ via the Hurewicz isomorphism. In particular, the following holds.

Lemma 6.14 *The isomorphism*

$$\Gamma_*(H_2(X)) \xrightarrow{\cong} H^*(\mathrm{Maps}(X, BSU(2)))$$

in theorem 4.10 is induced by the isomorphism

$$\overline{\overline{\mu}}_1 : H_2(X) \xrightarrow{\cong} H^2(\mathrm{Maps}(X, BSU(2))).$$

□

Remark: If M is a closed, simply-connected smooth four-manifold, we can use the map $\rho \colon C_1(M_0; FM) = FM_0 \to \mathrm{Maps}_1(M, BSU(2))$ and consider the composition

$$\rho^* \circ \mu_1 : H_2(M) \to H^2(FM_0).$$

Using the same methods as in [12], one can show that this composition is essentially Poincaré duality in the sense that the following diagram is commutative.

$$\begin{tikzcd}
H^2(M_0) \arrow[r, "\rho^*"] & H^2(FM_0) \\
H_2(M) \arrow[u, "Pd"] \arrow[r, "\mu_1"] & H^2(\mathrm{Maps}_1(M, BSU(2))) \arrow[u, "\rho^*"']
\end{tikzcd}$$

For $k, l \in \mathbb{N}$ define

$$\overline{\mu}_k \cup \overline{\mu}_l : H_u(Y) \otimes H_v(Y) \longrightarrow H^{4(k+l)-u-v}(\mathrm{Maps}_j(X, BSU(2)))$$

by $\overline{\mu}_k \cup \overline{\mu}_l(a \otimes b) = \overline{\mu}_k(a) \cup \overline{\mu}_l(b)$. The relation with the coproduct Δ on $H_*(Y)$ is the following.

Lemma 6.15 $\overline{\mu}_{k+l}(a) = \overline{\mu}_k \cup \overline{\mu}_l(\Delta(a))$.

Proof: It is obviously enough to prove the corresponding equality for $\tilde{\mu}$. But this follows immediately from the commutativity of the diagram

$$\begin{array}{ccc}
Y \times \mathrm{Maps}_j(Y, K(\mathbb{Z},4)) & \xrightarrow{\widetilde{ev}} & K(\mathbb{Z},4) \\
\downarrow \Delta & & \downarrow \Delta \\
Y \times \mathrm{Maps}_j(Y, K(\mathbb{Z},4)) \times Y \times \mathrm{Maps}_j(Y, K(\mathbb{Z},4)) & \xrightarrow{\widetilde{ev} \times \widetilde{ev}} & K(\mathbb{Z},4) \times K(\mathbb{Z},4)
\end{array}$$

together with the naturality properties of the slant product. □

Corollary 6.16

i) $\overline{\overline{\mu}}_n \equiv 0$ *for* $n \geq 2$.

ii) $\overline{\mu}_n \equiv 0$ *for* $n \geq 3$.

Proof: Notice that, since we are considering mapping spaces of basepoint preserving maps, $\overline{\mu}_n\big|_{H_0(Y)} \equiv 0$ for $n \geq 1$. Let $a \in H_2(Y)$ and $n \geq 2$. Then, with lemma 6.15, $\overline{\mu}_n(a) = \overline{\mu}_1 \cup \overline{\mu}_{n-1}(a \otimes 1 + 1 \otimes a) = 0$, which shows the first statement. Let $m \in H_4(Y)$. Then $\overline{\mu}_{n+1}(m) = \overline{\mu}_1 \cup \overline{\mu}_n(\Delta(m)) = 0$ according to the first statement, so the second statement is proved. □

Corollary 6.17 *Let* $Q \in H^4(\mathrm{Maps}(X, BSU(2))$ *be the intersection form of* Y *and* $[Y] \in H_4(Y)$ *the orientation class. Then* $\overline{\mu}_2([Y]) = 2 \cdot Q$.

Proof: Let $\alpha, \beta \in H_2(\mathrm{Maps}(X, BSU(2)))$. Then

$$\begin{aligned}
\langle \overline{\mu}_2([Y]), \alpha\beta \rangle &= \langle \overline{\mu}_1 \cup \overline{\mu}_1(\Delta([Y])), \alpha\beta \rangle \\
&= \langle \overline{\mu}_1 \otimes \overline{\mu}_1(\Delta([Y])), \alpha \otimes \beta + \beta \otimes \alpha \rangle \\
&= 2 \langle \overline{\mu}_1 \otimes \overline{\mu}_1(\Delta([Y])), \alpha \otimes \beta \rangle \\
&= 2 \langle \psi(\alpha) \otimes \psi(\beta), \Delta([Y]) \rangle \\
&= 2 \langle \psi(\alpha) \cup \psi(\beta), [Y] \rangle \\
&= 2 \cdot Q\big(\psi(\alpha), \psi(\beta)\big)
\end{aligned}$$

□

Remark: This implies that the map $\overline{\mu}_2$ is independent of the choice of j.

Now consider the map $\pi\colon \operatorname{Maps}(Y, S^4) \longrightarrow \operatorname{Maps}(Y, BSU(2))$ induced by the standard inclusion $i\colon S^4 \hookrightarrow BSU(2)$.

Lemma 6.18 *$im\ \mu_2 \subseteq ker\ \pi^*$*

Proof: The following diagram is commutative where the horizontal arrows represent evaluation maps.

$$\begin{tikzcd} Y \times \operatorname{Maps}(Y, S^4) \arrow[r] \arrow[d, "\mathrm{id}\times\pi"] & S^4 \arrow[d, "i"] \\ Y \times \operatorname{Maps}(Y, BSU(2)) \arrow[r] & BSU(2) \end{tikzcd}$$

Since $i^*(c^2) = 0 \in H^*(S^4)$, it follows that $im\ \mu_2 \subseteq ker\ \pi^*$. □

Now let M be a smooth, closed, simply-connected four-manifold and consider the map $\rho\colon C_j(M_0; FM) \longrightarrow \operatorname{Maps}_j(M, BSU(2))$ constructed in section 6.4.

Lemma 6.19 *For all $j \in \mathbf{N}$, $\rho^*(\mu_2([M])) = 0 \in H^4(C_j(M_0; FM))$.*

Proof: Recall that the map ρ factors through π. Hence it is enough to show that $\pi^*(\mu_2([M])) = 0$. But this follows from lemma 6.18. □

Theorem 6.20 *Let M be a smooth, closed, simply-connected four-manifold and let either $k = \mathbf{Z}/p\mathbf{Z}$ where $p \geq 5$ is a prime or $k \in \{\mathbf{Q}, \mathbf{Z}\}$. Then there is a non-zero class $2Q \in H^4(\operatorname{Maps}_j(M, BSU(2)); k)$ such that $2Q \in ker\ \rho^*$ for all $j \in \mathbf{N}$.*

Proof: In view of lemma 6.19, all we have to show is that $2 \cdot Q \neq 0 \in H^4(\operatorname{Maps}(M, BSU(2)); k)$. For $k = \mathbf{Z}/p\mathbf{Z}$, $p \geq 5$, this follows from theorem 5.21. For $K = \mathbf{Q}$, the result follows since $\Omega^2\mathcal{F}$ is rationally contractible, so $\mathrm{Maps}_j(M, BSU(2)) \to \mathrm{Maps}(X, BSU(2))$ induces an isomorphism in rational homology, where X denotes the 2-skeleton of M. For the integral case the following observation is due to G. Masbaum [19]. The map

$$H^*(\mathrm{Maps}(X, BSU(2)) \longrightarrow H^*(\mathrm{Maps}_j(M, BSU(2)))$$

is injective in integral cohomology, because in the Leray-Serre spectral sequence of the fibration

$$\Omega^3_j BSU(2) \longrightarrow \mathrm{Maps}_j(M, BSU(2)) \longrightarrow \mathrm{Maps}(X, BSU(2))$$

$E_2^{p,q}$ is all torsion if $q \neq 0$ and free if $q = 0$, so $E_*^{p,0}$ is not in the image of any differential. □

Remark: In $\mathbb{Z}/2\mathbb{Z}$ cohomology the class $2Q$ is obviously zero, which reconciles theorem 6.20 with theorem 6.11. Notice that, according to corollary 5.26, the class $2Q$ is zero in $\mathbb{Z}/3\mathbb{Z}$-homology. Our methods do not, however, enable us to decide, whether the map ρ^* is asymptotically injective in $\mathbb{Z}/3\mathbb{Z}$-homology.

The methods of section 6.4, at least for manifolds with even intersection forms, carry over to give in $\mathbb{Z}/p\mathbb{Z}$ homology an asymptotically surjective approximation of $\mathrm{Maps}(M, S^4)$ and, in the limit, the cokernel of the maps ρ_* is just the cokernel of the map π_*. We have seen that a way to construct elements in this cokernel (or in the kernel in cohomology) is via the map μ_2. We do not know, whether the class $2 \cdot Q \in H^4(\mathrm{Maps}(M, BSU(2)))$ generates the whole kernel of π^* as an ideal, but since the essential ingredient in our proof that $\pi^* \circ \mu_2 = 0$ is the fact that $c^2 \in H^*(BSU(2))$ maps to zero in $H^*(S^4)$, it is interesting to consider the inclusion $\mathbb{HP}^2 \hookrightarrow \mathbb{HP}^\infty$ and the induced map $\tau\colon \mathrm{Maps}(M, \mathbb{HP}^2) \to \mathrm{Maps}(M, \mathbb{HP}^\infty)$.

Theorem 6.21 *Let $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$. If Y has even intersection form then the map*

$$\tau\colon \operatorname{Maps}(Y, \mathbb{H}\mathrm{P}^2) \longrightarrow \operatorname{Maps}(Y, \mathbb{H}\mathrm{P}^\infty)$$

has a right inverse.

Proof: Choose a desuspension $X \simeq \Sigma W$. Then, using the standard homotopy equivalence $\Omega BSU(2) \simeq S^3$ and the inclusion $S^3 \to \Omega S^4$, one obtains a map

$$h\colon \operatorname{Maps}(X, BSU(2)) \longrightarrow \operatorname{Maps}(X, S^4)$$

which is a right homotopy inverse to the natural map given by composition with $i\colon S^4 \to BSU(2)$, and one checks, using similar methods as for lemma 4.14 and theorem 4.15, that there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Maps}(X, BSU(2)) \arrow[r] \arrow[d, "h"] & \Omega^2 S^6 \arrow[d, "\Omega^2 r"] \\
\operatorname{Maps}(X, S^4) \arrow[r, "\circ f"] \arrow[d] & \Omega^3 S^4 \arrow[d, dash] \\
\operatorname{Maps}(X, \mathbb{H}\mathrm{P}^2) \arrow[r, dash, "\circ f"] \arrow[d, dash] & \Omega^3 \mathbb{H}\mathrm{P}^2 \arrow[d] \\
\operatorname{Maps}(X, BSU(2)) \arrow[r, "\circ f"] & \Omega^3 BSU(2)
\end{tikzcd}$$

where $r\colon S^6 \to \Omega S^4$ is the adjoint of the Whitehead square $[\iota_4, \iota_4]$. From the fibration $S^{11} \hookrightarrow \mathbb{H}\mathrm{P}^2 \to \mathbb{H}\mathrm{P}^\infty$ we conclude that $\pi_7(\mathbb{H}\mathrm{P}^2) \cong \pi_7(BSU(2)) \cong \mathbb{Z}/12\mathbb{Z}$. Moreover, the composition

$$\Omega \mathbb{H}\mathrm{P}^2 \to \Omega BSU(2) \simeq S^3 \to \Omega S^4 \to \Omega \mathbb{H}\mathrm{P}^2$$

induces the identity on π_6. It follows that the composition

$$\Omega^2 S^6 \to \Omega^3 S^4 \to \Omega^3 \mathbb{H}\mathrm{P}^2$$

factors up to homotopy as

$$\Omega^2 S^6 \to \Omega^2 S^3 \to \Omega^3 S^4 \to \Omega^3 \mathbb{HP}^2$$

and we get maps of fibrations

$$\begin{tikzcd}
\mathrm{Maps}(Y, \mathbb{HP}^\infty) \arrow[r] \arrow[d] & \mathrm{Maps}(X, \mathbb{HP}^\infty) \arrow[r] \arrow[d] & \Omega^2 S^3 \arrow[d] \\
\mathrm{Maps}(Y, \mathbb{HP}^2) \arrow[r] \arrow[d] & \mathrm{Maps}(X, \mathbb{HP}^2) \arrow[r] \arrow[d] & \Omega^3 \mathbb{HP}^2 \arrow[d] \\
\mathrm{Maps}(Y, \mathbb{HP}^\infty) \arrow[r] & \mathrm{Maps}(X, \mathbb{HP}^\infty) \arrow[r] & \Omega^2 S^3
\end{tikzcd}$$

where all the compositions of vertical arrows are homotopy equivalences. □

Corollary 6.22 *Let $Y \simeq e^4 \cup_f X$, $X \simeq S^2 \vee \ldots \vee S^2$. Then for any prime p the map*

$$r_* : H_*(\mathrm{Maps}(Y, \mathbb{HP}^2); \mathbb{Z}/p\mathbb{Z}) \longrightarrow H_*(\mathrm{Maps}(Y, \mathbb{HP}^\infty); \mathbb{Z}/p\mathbb{Z})$$

is surjective.

Proof: For $p = 2$ this follows from corollary 5.18. For $p \geq 3$ we can without loss of generality assume that the intersection form of Y is even. Then the result follows from theorem 6.21. □

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